

The largest left quotient ring of a ring

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Abstract

The left quotient ring (i.e. the left classical ring of fractions) $Q_{cl}(R)$ of a ring R does not always exist and still, in general, there is no good understanding of the reason why this happens. In this paper, it is proved existence of the *largest left quotient ring* $Q_l(R)$, i.e. $Q_l(R) = S_0(R)^{-1}R$ where $S_0(R)$ is the *largest left regular denominator set* of R . It is proved that $Q_l(Q_l(R)) = Q_l(R)$; the ring $Q_l(R)$ is semi-simple iff $Q_{cl}(R)$ exists and is semi-simple; moreover, if the ring $Q_l(R)$ is left artinian then $Q_{cl}(R)$ exists and $Q_l(R) = Q_{cl}(R)$. The group of units $Q_l(R)^*$ of $Q_l(R)$ is equal to the set $\{s^{-1}t \mid s, t \in S_0(R)\}$ and $S_0(R) = R \cap Q_l(R)^*$. If there exists a finitely generated flat left R -module which is not projective then $Q_l(R)$ is not a semi-simple ring. We extend slightly Ore's method of localization to *localizable left Ore sets*, give a criterion of when a left Ore set is localizable, and prove that *all left and right Ore sets* of an arbitrary ring are localizable (not just denominator sets as in Ore's method of localization). Applications are given for certain classes of rings (semi-prime Goldie rings, Noetherian commutative rings, the algebra of polynomial integro-differential operators).

Key Words: the largest left quotient ring of a ring, the largest left (regular) denominator set of a ring, the classical left quotient ring of a ring, denominator set, the maximal left quotient rings of a ring.

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1 Introduction

The aim of the paper is to introduce the following new concepts and to prove their existence for an arbitrary ring: *the largest left quotient ring of a ring, the largest left (regular) denominator set of a ring, the maximal left quotient rings of a ring, the largest (two-sided) quotient ring of a ring, the maximal (two-sided) quotient rings of a ring, a (left) localization maximal ring.*

Throughout, module means a left module; R is a ring with 1; \mathcal{C}_R is the set of (left and right) regular elements of the ring R (i.e. \mathcal{C}_R is the set of non-zero-divisors of R); $Q_{cl}(R)$ is the left quotient ring of R (if exists); $\text{Den}_l(R, 0)$ is the set of regular left denominator sets S in R ($S \subseteq \mathcal{C}_R$).

The largest left regular denominator set and the largest left quotient ring of a ring.

- (Theorem 2.1) *There exists the largest (w.r.t. inclusion) left regular denominators set $S_0(R)$ in R and so $Q_l(R) := S_0(R)^{-1}R$ is the largest left quotient ring of R .*
- (Corollary 2.5) *The set $\text{Den}_l(R, 0)$ of left regular denominator sets of R is a complete lattice and an abelian monoid.*

The next theorem describes the group of units $Q_l(R)^*$ of the ring $Q_l(R)$ and its connection with $S_0(R)$.

• (Theorem 2.8)

1. $S_0(Q_l(R)) = Q_l(R)^*$ and $S_0(Q_l(R)) \cap R = S_0(R)$.
2. $Q_l(R)^* = \langle S_0(R), S_0(R)^{-1} \rangle$, i.e. the group of units of the ring $Q_l(R)$ is generated by the sets $S_0(R)$ and $S_0(R)^{-1} := \{s^{-1} \mid s \in S_0(R)\}$.
3. $Q_l(R)^* = \{s^{-1}t \mid s, t \in S_0(R)\}$.
4. $Q_l(Q_l(R)) = Q_l(R)$.

The next theorem gives an answer to the question of when the ring $Q_l(R)$ is semi-simple (Theorem 2.8 is a key step in proving Theorem 2.9, statements 2-5 is Goldie's Theorem).

• (Theorem 2.9) *The following statements are equivalent.*

1. $Q_l(R)$ is a semi-simple ring.
2. $Q_{cl}(R)$ exists and is a semi-simple ring.
3. R is a left order in a semi-simple ring.
4. R has finite left rank, satisfies the ascending chain condition on left annihilators, and is a semi-prime ring.
5. A left ideal of the ring R is essential iff it contains a regular element.

If one of these conditions hold then $S_0(R) = \mathcal{C}_R$ and $Q_l(R) = Q_{cl}(R)$.

The next result is a sufficient condition for the ring $Q_l(R)$ not being a semi-simple ring.

• (Corollary 2.12) *If there exists a finitely generated flat R -module which is not projective then the ring $Q_l(R)$ is not semi-simple.*

The next corollary gives a sufficient condition for existing of $Q_{cl}(R)$.

• (Corollary 2.10) *If $Q_l(R)$ is a left artinian ring then $S_0(R) = \mathcal{C}_R$ and $Q_l(R) = Q_{cl}(R)$.*

Let $S_0^r(R)$ and $Q_r(R) := RS_0^r(R)^{-1}$ be the *largest regular right denominator set* in R and the *largest right quotient ring* of R , respectively. In general, the following natural questions have negative answers as the algebra $\mathbb{I}_1 := K\langle x, \frac{d}{dx}, \int \rangle$ of polynomial integro-differential operators over a field K of characteristic zero demonstrates [2]:

Question 1. Is $S_0(R) = S_0^r(R)$?

Question 2. Is $S_0(R) \subseteq S_0^r(R)$ or $S_0(R) \supseteq S_0^r(R)$?

Question 3. Are the rings $Q_l(R)$ and $Q_r(R)$ isomorphic?

Though, for the algebra \mathbb{I}_1 the next question has positive answer [2].

Question 4. Are the rings $Q_l(R)$ and $Q_r(R)$ anti-isomorphic?

Remark. The algebra \mathbb{I}_1 is neither left nor right Noetherian, it contains infinite direct sums of nonzero left (and right) ideals, neither left nor right quotient ring of \mathbb{I}_1 exists [2].

Notation:

- $\text{Ore}_l(R) := \{S \mid S \text{ is a left Ore set in } R\};$
- $\text{Den}_l(R) := \{S \mid S \text{ is a left denominator set in } R\};$
- $\text{Loc}_l(R) := \{S^{-1}R \mid S \in \text{Den}_l(R)\};$
- $\text{Ass}_l(R) := \{\text{ass}(S) \mid S \in \text{Den}_l(R)\}$ where $\text{ass}(S) := \{r \in R \mid sr = 0 \text{ for some } s = s(r) \in S\};$

- $\text{Den}_l(R, \mathfrak{a}) := \{S \in \text{Den}_l(R) \mid \text{ass}(S) = \mathfrak{a}\}$ where $\mathfrak{a} \in \text{Ass}_l(R)$;
- $S_{\mathfrak{a}} = S_{\mathfrak{a}}(R) = S_{l,\mathfrak{a}}(R)$ is the *largest element* of the poset $(\text{Den}_l(R, \mathfrak{a}), \subseteq)$ and $Q_{\mathfrak{a}}(R) := Q_{l,\mathfrak{a}}(R) := S_{\mathfrak{a}}^{-1}R$ is the *largest left quotient ring associated to \mathfrak{a}* , $S_{\mathfrak{a}}$ exists (Theorem 2.1.(2));
- In particular, $S_0 = S_0(R) = S_{l,0}(R)$ is the largest element of the poset $(\text{Den}_l(R, 0), \subseteq)$ and $Q_l(R) := S_0^{-1}R$ is the largest left quotient ring of R ;
- $\text{Loc}_l(R, \mathfrak{a}) := \{S^{-1}R \mid S \in \text{Den}_l(R, \mathfrak{a})\}$.

For each denominator set $S \in \text{Den}_l(R, \mathfrak{a})$ where $\mathfrak{a} := \text{ass}(S)$, there are natural ring homomorphisms $R \xrightarrow{\pi} R/\mathfrak{a} \rightarrow S^{-1}R$. In Section 3, connections between the denominator sets $\text{Den}_l(R, \mathfrak{a})$, $\text{Den}_l(R/\mathfrak{a}, 0)$ and $\text{Den}_l(S^{-1}R, 0)$ are established (Lemma 3.2, Lemma 3.3 and Proposition 3.4; these results are too technical to explain in the Introduction). They are used to prove the following results.

- (Lemma 3.3.(2)) *Let $a \in \text{Ass}_l(R)$ and $\pi : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$. Then $\pi^{-1}(S_0(R/\mathfrak{a})) = S_{\mathfrak{a}}(R)$, $\pi(S_{\mathfrak{a}}(R)) = S_0(R/\mathfrak{a})$ and $Q_{\mathfrak{a}}(R) = Q_l(R/\mathfrak{a})$.*
- (Lemma 3.7.(2)) *The set $\text{max.Den}_l(R)$ of maximal elements of the poset $(\text{Den}_l(R), \subseteq)$ of left denominator sets of an arbitrary ring R is a non-empty set.*

This means that the set $\text{max.Loc}_l(R)$ of maximal left quotient rings of an arbitrary ring R is a non-empty set, and $\text{max.Loc}_l(R) = \{S^{-1}R \mid S \in \text{max.Den}_l(R)\}$.

Example. $\text{max.Den}_l(\mathbb{I}_1) = \{\mathbb{I}_1 \setminus F\}$ (Proposition 5.8.(3)) where F is the only proper ideal of the algebra of polynomial integro-differential operators \mathbb{I}_1 and $\text{max.Loc}_l(\mathbb{I}_1) = \{\text{Frac}(A_1)\}$ (Proposition 5.8.(2)) where $\text{Frac}(A_1)$ is the *Weyl skew field*, i.e. the classical (left and right) ring of fractions of the first Weyl algebra $A_1 = K\langle x, \frac{x}{dx} \rangle$.

A new class of rings is introduced, the class of *left localization maximal rings*, see Section 3 (intuitively, one can invert nothing more in these rings). For an arbitrary ring R , a criterion is given in terms of this class of rings of when a left localization $S^{-1}R$ of R is a maximal left localization of R .

- (Theorem 3.11) *Let a ring A be a left localization of a ring R , i.e. $A \in \text{Loc}_l(R, \mathfrak{a})$ for some $\mathfrak{a} \in \text{Ass}_l(R)$. Then $A \in \text{max.Loc}_l(R)$ iff $Q_l(A) = A$ and $\text{Ass}_l(A) = \{0\}$, i.e. A is a left localization maximal ring.*

Example. Let V be an infinite dimensional vector space with countable basis over a field K , let $\mathcal{C} := \{\varphi \in \text{End}_K(V) \mid \dim_K(\text{im}(\varphi)) < \infty\}$ be the ideal of compact operators/linear maps of the algebra $\text{End}_K(V)$. Then the factor algebra $\text{End}_K(V)/\mathcal{C}$ is a left localization maximal ring (Theorem 5.2.(5)). Moreover, the set \mathcal{F} of Fredholm linear maps in V is the maximal left (resp. right; left and right) denominator set of the ring $\text{End}_K(V)$ and

$$\mathcal{F}^{-1}\text{End}_K(V) \simeq \text{End}_K(V)\mathcal{F}^{-1} \simeq \text{End}_K(V)/\mathcal{C}$$

is the maximal left (resp. right; left and right) localization ring of $\text{End}_K(V)$ (Theorem 5.2).

A slight extension/generalization of Ore's method of localization. Ore's method of left localization says that we can localize a ring R *precisely* at the left denominator sets $\text{Den}_l(R)$ of the ring R . Notice that each left denominator set is a left Ore set but not the other way round, in general. We introduce the concept of a *localizable left Ore set* and give a criterion of when a left Ore set is localizable (Theorem 3.15). Each left denominator set is a localizable left Ore set but not vice versa, in general. We extend Ore's method of localization to localizable left Ore sets and prove an analogue of Ore's Theorem (by using Ore's Theorem) for localizable left Ore sets.

- (Corollary 3.16) *Let S be a localizable left Ore set in a ring R . Then there exists an ordered pair (Q, f) where Q is a ring and $f : R \rightarrow Q$ is a ring homomorphism such that*
(i) for all $s \in S$, $f(s)$ is a unit in Q ;
and if (Q', f') is another pair satisfying the condition (i) then there is a unique ring homomorphism $h : Q \rightarrow Q'$ such that $f' = hf$. The ring Q is unique up to isomorphism. The ring Q is isomorphic to the left localization of the ring $R/\mathfrak{p}(S)$ at the left denominator set $\pi(S) \in \text{Den}_l(R/\mathfrak{p}(S), 0)$ where the ideal $\mathfrak{p}(S)$ of the ring R is defined in (11) and $\pi : R \rightarrow R/\mathfrak{p}(S)$, $a \mapsto a + \mathfrak{p}(S)$.

In Section 4, we prove *two-sided* analogues of some of the results of Sections 2 and 3. In most cases the proofs are easy corollaries of the corresponding one-sided (i.e. left and right) results but there are some surprises. In particular,

- (Theorem 4.15) *Every (left and right) Ore set is localizable.*

Therefore we can localize at *all* the (left and right) Ore sets not just at the (left and right) denominator sets as in Ore's method of localization.

- (Corollary 4.16) *Let S be an Ore set in a ring R . Then there exists an ordered pair (Q, f) where Q is a ring and $f : R \rightarrow Q$ is a ring homomorphism such that*
(i) for all $s \in S$, $f(s)$ is a unit in Q ;
and if (Q', f') is another pair satisfying the condition (i) then there is a unique ring homomorphism $h : Q \rightarrow Q'$ such that $f' = hf$. The ring Q is unique up to isomorphism. The ring Q is isomorphic to the localization of the ring $R/\mathfrak{p}(S)$ at the denominator set $\pi(S) \in \text{Den}(R/\mathfrak{p}(S), 0)$ where the ideal $\mathfrak{p}(S)$ of the ring R is defined in (19) and $\pi : R \rightarrow R/\mathfrak{p}(S)$, $a \mapsto a + \mathfrak{p}(S)$.

In Section 5, the largest (left; right; left and right) and maximal (left; right; left and right) quotient rings are found for following rings: the endomorphism algebra $\text{End}_K(V)$ of an infinite dimensional vector space V with countable basis, semi-prime Goldie rings, the algebra \mathbb{I}_1 of polynomial integro-differential operators and Noetherian commutative rings.

2 The largest left quotient ring of a ring

In this section, existence of the largest left quotient ring of a ring is proved (Theorem 2.1). Proofs of Theorems 2.8 and 2.9 are given.

The largest left quotient ring of a ring. Let R be a ring. A *multiplicatively closed subset* S of R (i.e. a multiplicative sub-semigroup of (R, \cdot) such that $1 \in S$ and $0 \notin S$) is said to be a *left Ore set* if it satisfies the *left Ore condition*: for each $r \in R$ and $s \in S$, $Sr \cap Rs \neq \emptyset$. Let S be a (non-empty) multiplicatively closed subset of R , and let $\text{ass}(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$ (if, in addition, S is a left Ore set then $\text{ass}(S)$ is an ideal of the ring R). Then a *left quotient ring* of R with respect to S (a *left localization* of R at S) is a ring Q together with a homomorphism $\varphi : R \rightarrow Q$ such that

- (i) for all $s \in S$, $\varphi(s)$ is a unit of Q ,
- (ii) for all $q \in Q$, $q = \varphi(s)^{-1}\varphi(r)$ for some $r \in R$, $s \in S$, and
- (iii) $\ker(\varphi) = \text{ass}(S)$.

If exists the ring Q is unique up to isomorphism, usually it is denoted by $S^{-1}R$. Recall that $S^{-1}R$ exists iff S is a left Ore set and the set $\overline{S} = \{s + \text{ass}(S) \in R/\text{ass}(S) \mid s \in S\}$ consists of regular elements ([5], 2.1.12). If the last two conditions are satisfied then S is called a *left denominator set*. Similarly, a *right Ore set*, the *right Ore condition*, the *right denominator set* and the *right quotient ring* RS^{-1} are defined. If both $S^{-1}R$ and RS^{-1} exist then they are isomorphic ([5], 2.1.4.(ii)). The left quotient ring of R with respect to the set \mathcal{C}_R of all regular elements is

called the *left quotient ring* of R , if exists, it is denoted by $\text{Frac}_l(R)$ or $Q_{cl}(R)$. Similarly, the *right quotient ring*, $\text{Frac}_r(R) = Q_{cl}^r(R)$, is defined. If both left and right quotient rings of R exist then they are isomorphic and we write simply $\text{Frac}(R)$ or $Q(R)$ in this case.

Theorem 2.1 1. For each $\mathfrak{a} \in \text{Ass}_l(R)$, the set $\text{Den}_l(R, \mathfrak{a})$ is an ordered abelian semigroup ($S_1 S_2 = S_2 S_1$, and $S_1 \subseteq S_2$ implies $S_1 S_3 \subseteq S_2 S_3$) where the product $S_1 S_2 = \langle S_1, S_2 \rangle$ is the multiplicative subsemigroup of (R, \cdot) generated by S_1 and S_2 .

2. $S_{\mathfrak{a}} := S_{\mathfrak{a}}(R) := \bigcup_{S \in \text{Den}_l(R, \mathfrak{a})} S$ is the largest element (w.r.t. \subseteq) in $\text{Den}_l(R, \mathfrak{a})$. The set $S_{\mathfrak{a}}$ is called the largest left denominator set associated to \mathfrak{a} .

3. Let $S_i \in \text{Den}_l(R, \mathfrak{a})$, $i \in I$, where I is an arbitrary non-empty set. Then

$$\langle S_i \mid i \in I \rangle := \bigcup_{\emptyset \neq J \subseteq I, |J| < \infty} \prod_{j \in J} S_j \in \text{Den}_l(R, \mathfrak{a}) \quad (1)$$

the left denominator set generated by the left denominators sets S_i , it is the least upper bound of the set $\{S_i\}_{i \in I}$ in $\text{Den}_l(R, \mathfrak{a})$, i.e. $\langle S_i \mid i \in I \rangle = \bigvee_{i \in I} S_i$.

Remark. Clearly, $S_1 S_2 = S_1 \vee S_2$, the join of S_1 and S_2 .

Proof. 1. It remains to show that $S_1 S_2 \in \text{Den}_l(R, \mathfrak{a})$, that is, $S_1 S_2$ is a left Ore set in R with $\text{ass}(S_1 S_2) = \mathfrak{a}$ and $rs = 0$ where $r \in R$ and $s \in S$ implies $tr = 0$ for some $t \in S$. To prove that the left Ore condition holds for the multiplicatively closed set $S = S_1 S_2$ we have to show that, for each $s \in S$ and $a \in R$, there exist elements $t \in S$ and $b \in R$ such that $ta = bs$. We use induction on the length $l(s) = \min\{n \mid s = s_1 \cdots s_n \text{ where all } s_i \in S_1 \cup S_2\}$ of the element s . If $l(s) = 1$, i.e. $s \in S_1 \cup S_2$, then the statement is obvious since S_1 and S_2 are left Ore sets. Suppose that $n := l(s) > 1$, and the statement holds for $n' < n$. Fix a presentation $s = s_1 \cdots s_{n-1} s_n$ where all $s_i \in S_1 \cup S_2$. Then $t_1 a = b_1 s_n$ for some elements $t_1 \in S_1 \cup S_2$ and $b_1 \in R$, and, by induction, $t_2 b_1 = b s_1 \cdots s_{n-1}$ for some elements $t_2 \in S$ and $b \in R$. Clearly, $t = t_2 t_1 \in S$ and $ta = t_2(t_1 a) = (t_2 b_1) s_n = bs$, as required.

Let us show that $\text{ass}(S) = \mathfrak{a}$. Clearly, $\text{ass}(S_1 S_2) \supseteq \mathfrak{a}$. Let $r \in \text{ass}(S_1 S_2)$. Then $sr = 0$ for some element $s = s_1 \cdots s_m \in S$ where all $s_i \in S_1 \cup S_2$. Without loss of generality we may assume that $s_{\text{odd}} \in S_1$ and $s_{\text{even}} \in S_2$. Then $s_2 \cdots s_m r \in \mathfrak{a}$, and so there exists $s'_2 \in S_2$ such that $(s'_2 s_2) s_3 \cdots s_m r = s''_2 s_3 \cdots s_m r = 0$ where $s''_2 = s'_2 s_2 \in S_2$. By induction on m , we conclude that $r \in \mathfrak{a}$. Therefore, $\text{ass}(S) = \mathfrak{a}$.

Finally, suppose that $rs = 0$ where $r \in R$ and $s \in S$. We have to show that $tr = 0$ for some $t \in S$. Fix a presentation $s = s_1 \cdots s_n$ where $s_i \in S_1 \cup S_2$. Then $0 = rs = r s_1 \cdots s_n$ implies that $t_1 r s_1 \cdots s_{n-1} = 0$ for some $t_1 \in S_1 \cup S_2$ since $S_1, S_2 \in \text{Den}_l(R, \mathfrak{a})$. Similarly, $t_2 t_1 r s_1 \cdots s_{n-2} = 0$ for some $t_2 \in S_1 \cup S_2$. Repeating the same argument several times we see that $t_n t_{n-1} \cdots t_1 r = 0$ for some $t_i \in S_1 \cup S_2$. Notice that $t := t_n t_{n-1} \cdots t_1 \in S$ and $tr = 0$, as required.

2. Let $s_1, s_2 \in S_{\mathfrak{a}}$. Then $s_1 \in S_1$ and $s_2 \in S_2$ for some $S_1, S_2 \in \text{Den}_l(R, \mathfrak{a})$, and so $s_1, s_2 \in S_1 S_2 \in \text{Den}_l(R, \mathfrak{a})$, by statement 1. Therefore, $S_{\mathfrak{a}} \in \text{Den}_l(R, \mathfrak{a})$, and so $S_{\mathfrak{a}}$ is the largest element in $\text{Den}_l(R, \mathfrak{a})$.

3. Statement 3 follows from statement 1. \square

Definition. For each ideal $\mathfrak{a} \in \text{Ass}_l(R)$, the ring $Q_{\mathfrak{a}}(R) := S_{\mathfrak{a}}^{-1} R$ is called the *largest left quotient ring associated to \mathfrak{a}* . When $\mathfrak{a} = 0$, the ring $Q_l(R) := Q_{\mathfrak{a}}(R) := S_0^{-1} R$ is called the *largest left quotient ring of R* and $S_0 = S_0(R)$ is called the *largest left regular denominator set of R* .

The next obvious corollary shows that $Q_l(R)$ is a generalization of the classical left quotient ring $Q_{cl}(R)$.

Corollary 2.2 1. If the classical left quotient ring $Q_{cl}(R) := C_R^{-1} R$ exists then the set of regular elements C_R of the ring R is the largest left regular denominator set and $Q_l(R) = Q_{cl}(R)$.

2. Let R_1, \dots, R_n be rings. Then $Q_l(\prod_{i=1}^n R_i) \simeq \prod_{i=1}^n Q_l(R_i)$.

Proof. It is obvious. \square

Question 2.3 Can a ring monomorphism $f : A \rightarrow B$ be lifted (necessarily uniquely) to a ring monomorphism $f' : Q_l(A) \rightarrow Q_l(B)$?

In general, the answer is no.

Proposition 2.4 (Corollary 9.9, [2]) Let K be a field of characteristic zero and $A_1 = K\langle x, \frac{d}{dx} \rangle$ be the ring of polynomial differential operators (the first Weyl algebra). Then the inclusion $A_1 \rightarrow \mathbb{I}_1$ cannot be lifted neither to a ring homomorphism $\text{Frac}(A_1) = Q_l(A_1) \rightarrow Q_l(\mathbb{I}_1)$ nor to $\text{Frac}(A_1) = Q_r(A_1) \rightarrow Q_r(\mathbb{I}_1)$.

Let (P, \leq) be a *partially ordered set*, a *poset*, for short ($a \leq a$ for all $a \in P$; $a \leq b$ and $b \leq a$ implies $a = b$; $a \leq b$ and $b \leq c$ implies $a \leq c$). For a subset S of P , an element $x \in P$ is called an *upper bound* (resp. a *lower bound*) if $s \leq x$ (resp. $s \geq x$) for all $s \in S$. The *least upper bound* for S is the least element in the set of all upper bounds for S . Similarly, the *greatest lower bound* for S is defined. A *lattice* is a poset such that each pair x, y of elements of the set S has both the least upper bound $x \vee y$ (called the *join* for x and y) and the greatest lower bound $x \wedge y$ (called the *meet* of x and y). It follows then by induction that every non-empty finite set of elements has the join and the meet. A lattice L is *complete* if every subset S of L has the *least upper bound* (the *join* of S) denoted $\sup(S)$ or $\bigvee_{s \in S} s$ and the *greatest lower bound* (the *meet* of S) denoted $\inf(S)$ or $\bigwedge_{s \in S} s$. In a complete lattice there exists the greatest element $\sup(L)$ denoted by 1, and the smallest element $\inf(L)$ denoted by 0. By definition, $\sup(\emptyset) = 0$. Let (L, \leq) be a lattice and $a, b \in L$ with $a \leq b$. The set $[a, b] := \{x \in L \mid a \leq x \leq b\}$ is the *interval* between a and b . The interval $[a, b]$ is a sublattice of L with $\inf([a, b]) = a$ and $\sup([a, b]) = b$.

Corollary 2.5 The abelian monoid $\text{Den}_l(R, 0)$ is a complete lattice such that $S_1 S_2 = S_1 \vee S_2$ and $\bigwedge_{i \in I} S_i = \bigvee_{S \in \text{Den}_l(R, 0), S \subseteq \bigcap_{i \in I} S_i} S_i$ where all $S_i \in \text{Den}_l(R, 0)$.

Proof. The largest and least elements of the poset $\text{Den}_l(R, 0)$ are $S_0(R)$ and $\{1\}$ respectively. Clearly, $\{1\}$ is the identity element of the semigroup $\text{Den}_l(R, 0)$. By Theorem 2.1.(3), each nonempty subset of $\text{Den}_l(R, 0)$ has the least upper bound, hence $\text{Den}_l(R, 0)$ is a complete lattice by Proposition 1.2, Sect. 3, [6] and $\bigwedge_{i \in I} S_i = \bigvee_{S \in \text{Den}_l(R, 0), S \subseteq \bigcap_{i \in I} S_i} S_i$. \square

Clearly, $\bigwedge_{i \in I} S_i$ is the largest element of the set $\{S \mid S \in \text{Den}_l(R, 0), S \subseteq \bigcap_{i \in I} S_i\}$.

Corollary 2.6 1. Let R be a ring. Each ring automorphism $\sigma \in \text{Aut}(R)$ of the ring R has the unique extension $\sigma \in \text{Aut}(Q_l(R))$ to an automorphism of the ring $Q_l(R)$ given by the rule $\sigma(s^{-1}r) = \sigma(s)^{-1}\sigma(r)$ where $s \in S_0(R)$ and $r \in R$.

2. The group $\text{Aut}(R)$ is a subgroup of the group $\text{Aut}(Q_l(R))$. Moreover, $\text{Aut}(R) = \{\tau \in \text{Aut}(Q_l(R)) \mid \tau(R) = R\}$.

Proof. 1. By the uniqueness of the set $S_0(R)$, $\sigma(S_0(R)) = S_0(R)$ for all elements $\sigma \in \text{Aut}(R)$. Now, statement 1 follows from the universal property of the localization at $S_0(R)$.

2. Statement 2 follows from statement 1. \square

For a ring R , let R^* be its group of units and $\text{Inn}(R) := \{\omega_u \mid u \in R^*\}$ be the group of inner automorphisms of the ring R where $\omega_u(r) = uru^{-1}$ is the inner automorphism determined by the element u . The next proposition is used in the proof of Theorem 2.8.

Proposition 2.7 Let R be a ring, $S \in \text{Den}_l(R, 0)$, and $T \in \text{Den}_l(S^{-1}R, 0)$; and so $R \subseteq S^{-1}R \subseteq T^{-1}(S^{-1}R)$ are natural inclusions of rings. Then

1. $T^{-1}(S^{-1}R) = T_1^{-1}(S^{-1}R)$ for some $T_1 \in \text{Den}_l(S^{-1}R, 0)$ such that $S, S^{-1} \subseteq T_1$; in particular, $sT_1s^{-1} = T_1$ for all $s \in S$.

2. If, in addition, $S \subseteq T$ then $T \cap R \in \text{Den}_l(R, 0)$ and $S \subseteq T \cap R$.

Proof. 1. For each $s \in S$, $T^{-1}(S^{-1}R) = \omega_s(T^{-1}(S^{-1}R)) = \omega_s(T)^{-1}(S^{-1}R)$ and $\omega_s(T) \in \text{Den}_l(S^{-1}R, 0)$. It suffices to take $T_1 := \langle S, S^{-1}, \omega_s(T) \mid s \in S \rangle$ in $\text{Den}_l(S^{-1}R, 0)$ since clearly $S, S^{-1} \in \text{Den}_l(S^{-1}R, 0)$ and, for each non-empty finite subset J of S , $(\prod_{s \in J} \omega_s(T))^{-1}(S^{-1}R) = T^{-1}(S^{-1}R)$. In more detail,

$$T_1^{-1}(S^{-1}R) = \bigcup_{\emptyset \neq J \subseteq I, |J| < \infty} \left(\prod_{s \in J} \omega_s(T) \right)^{-1}(S^{-1}R) = \bigcup_{\emptyset \neq J \subseteq I, |J| < \infty} T^{-1}(S^{-1}R) = T^{-1}(S^{-1}R).$$

2. The set $T' := T \cap R$ is a multiplicatively closed subset of \mathcal{C}_R that contains the set S . It remains to show that the left Ore condition holds for T' in R : for each elements $t' \in T'$ and $r \in R$, $T'r \cap Rt' \neq \emptyset$. Since $T \in \text{Den}_l(S^{-1}R, 0)$, $Tr \cap S^{-1}Rt' \neq \emptyset$. Take an element, say u , from the intersection. The element u can be written in two different ways as follows

$$s_1^{-1}t_1 \cdot r = s_2^{-1}r_2t'$$

for some $s_1^{-1}t_1 \in T$ and $s_2^{-1}r_2 \in S^{-1}R$ where $s_1, s_2 \in S$ and $t_1, r_2 \in R$. Clearly, $t_1 = s_1 \cdot s_1^{-1}t_1 \in R \cap T = T'$ (since $S \subseteq T$), and $s_1s_2^{-1} = s_3^{-1}r_3$ for some $s_3 \in S$ and $r_3 \in R$. Then the element

$$s_3t_1 \cdot r = s_3s_1 \cdot s_1^{-1}t_1r = s_3s_1s_2^{-1}r_2t' = r_3r_2t'$$

belongs to the intersection $Tr \cap Rt'$ since $s_3t_1 \in T$ (since $S \subseteq T$) and $r_3r_2 \in R$. \square

The group of units $Q_l(R)^*$ of $Q_l(R)$. For a ring R and its largest left quotient ring $Q_l(R)$, Theorem 2.8 is used in the proof of Theorem 2.9 and gives an answers to the following natural questions:

- What is $S_0(Q_l(R))$?
- What is $S_0(Q_l(R)) \cap R$?
- What is the group $Q_l(R)^*$ of units of the ring $Q_l(R)$?
- Is the natural inclusion $Q_l(R) \subseteq Q_l(Q_l(R))$ an equality?

Theorem 2.8 1. $S_0(Q_l(R)) = Q_l(R)^*$ and $S_0(Q_l(R)) \cap R = S_0(R)$.

2. $Q_l(R)^* = \langle S_0(R), S_0(R)^{-1} \rangle$, i.e. the group of units of the ring $Q_l(R)$ is generated by the sets $S_0(R)$ and $S_0(R)^{-1} := \{s^{-1} \mid s \in S_0(R)\}$.

3. $Q_l(R)^* = \{s^{-1}t \mid s, t \in S_0(R)\}$.

4. $Q_l(Q_l(R)) = Q_l(R)$.

Proof. 1–3. It is obvious that $G := \langle S_0(R), S_0(R)^{-1} \rangle \subseteq Q_l(R)^* \subseteq S_0(Q_l(R))$. Applying Proposition 2.7.(2) in the situation where $S = S_0(R)$ and $T = S_0(Q_l(R))$ we see that

$$S_0(R) \subseteq T' := S_0(Q_l(R)) \cap R \in \text{Den}_l(R, 0),$$

and so $S_0(R) = T'$, by the maximality of $S_0(R)$. Let $q \in S_0(Q_l(R))$. Then $q = s^{-1}t$ for some elements $s \in S_0(R)$ and $t = sq \in S_0(Q_l(R)) \cap R = S_0(R)$. Therefore, $S_0(Q_l(R)) \subseteq \{s^{-1}t \mid s, t \in S_0(R)\} \subseteq G$, and so $G = Q_l(R)^* = S_0(Q_l(R)) = \{s^{-1}t \mid s, t \in S_0(R)\}$. This proves statements 1–3.

4. Statement 4 follows from statement 1. \square

Necessary and sufficient conditions for $Q_l(R)$ to be a semi-simple ring. A ring Q is called a *ring of quotients* if every element $c \in \mathcal{C}_Q$ is invertible. A subring R of a ring of quotients Q is called a *left order* in Q if \mathcal{C}_R is a left Ore set and $\mathcal{C}_R^{-1}R = Q$. A ring R has *finite left rank* (i.e. *finite left uniform dimension*) if there are no infinite direct sums of nonzero left ideals in R .

The next theorem gives an answer to the question of when $Q_l(R)$ is a semi-simple ring.

Theorem 2.9 *The following properties of a ring R are equivalent.*

1. $Q_l(R)$ is a semi-simple ring.
2. $Q_{cl}(R)$ exists and is a semi-simple ring.
3. R is a left order in a semi-simple ring.
4. R has finite left rank, satisfies the ascending chain condition on left annihilators and is a semi-prime ring.
5. A left ideal of R is essential iff it contains a regular element.

If one of the equivalent conditions hold then $S_0(R) = \mathcal{C}_R$ and $Q_l(R) = Q_{cl}(R)$.

Proof. Goldie's Theorem states that $2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$.

(3 \Rightarrow 1) If statement 3 holds then $Q_l(R) = \mathcal{C}_R^{-1}R$ is a semi-simple ring.

(1 \Rightarrow 3) We have to show that \mathcal{C}_R is a left Ore set since then $\mathcal{C}_R = S_0(R)$ and $Q_l(R) = Q_{cl}(R)$ is a semi-simple ring. Notice that $S_0(R) \subseteq \mathcal{C}_R$. Let $q \in \mathcal{C}_R$. Then the $Q_l(R)$ -module monomorphism $\cdot q : Q_l(R) \rightarrow Q_l(R)$, $x \mapsto xq$, is an isomorphism, and its inverse is necessarily of the form $\cdot p$ for some element $p \in Q_l(R)$. Clearly, $pq = 1$ and $qp = 1$, i.e. $q \in Q_l(R)^*$. By Theorem 2.8.(3), $q = s^{-1}t$ for some elements $s, t \in S_0(R)$. We have to show that, for each $q \in \mathcal{C}_R$ and $r \in R$, there exists an element $c \in \mathcal{C}_R$ such that $c r q^{-1} = c r t^{-1} s \in R$. Since $t \in S_0(R)$, there exists an element $s_1 \in S_0(R)$ such that $s_1 r t^{-1} \in R$. It suffices to take $c = s_1$ since $S_0(R) \subseteq \mathcal{C}_R$. \square

The next corollary gives an interesting criterion of when the classical quotient ring $Q_{cl}(R) = \mathcal{C}_R^{-1}R$ exists.

Corollary 2.10 *If the ring $Q_l(R)$ is a left artinian ring then $S_0(R) = \mathcal{C}_R$ and $Q_{cl}(R) = Q_l(R)$.*

Proof. Each left artinian ring is left Noetherian, hence the left $Q_l(R)$ -module $Q_l(R)$ has finite length. Now, we can repeat the proof of the implication $1 \Rightarrow 3$ of Theorem 2.9, where, in fact, we only used the fact that the left $Q_l(R)$ -module $Q_l(R)$ has finite length. \square

Proposition 2.11 (Proposition 11.6, [6]; [4]) *Let A be a subring of a ring B . If M is a finitely generated flat A -module such that $B \otimes_A M$ is a projective B -module then M is a projective A -module.*

Corollary 2.12 *If there exists a finitely generated flat R -module M which is not projective then the ring $Q_l(R)$ is not a semi-simple ring.*

Proof. If $Q_l(R)$ were a semi-simple ring then $Q_l(R) \otimes_R M$ would be a projective $Q_l(R)$ -module, and so M would be a projective R -module, by Proposition 2.11, a contradiction. \square

3 The maximal left quotient rings of a ring

In this section, a new class of rings, the class of left localization maximal rings, is introduced. It is proved that, for an arbitrary ring R , the set of maximal elements of the poset $(\text{Den}_l(R), \subseteq)$ is a non-empty set (Lemma 3.7.(2)), and therefore the set of maximal left quotient rings of the ring R is a non-empty set. A criterion is given (Theorem 3.11) for a left quotient ring of a ring to be a maximal left quotient ring of the ring. Many results on denominator sets are proved. In particular, for each denominator set $S \in \text{Den}_l(R, \mathfrak{a})$, connections are established between the left denominator sets $\text{Den}_l(R, \mathfrak{a})$, $\text{Den}_l(R/\mathfrak{a}, 0)$ and $\text{Den}_l(S^{-1}R, 0)$.

Proposition 3.1 1. *For each ring $A \in \text{Loc}_l(R, \mathfrak{a})$ where $\mathfrak{a} \in \text{Ass}_l(R)$, the set $\text{Den}_l(R, \mathfrak{a}, A) := \{S \in \text{Den}_l(R, \mathfrak{a}) \mid S^{-1}R = A\}$ is an ordered submonoid of $\text{Den}_l(R, \mathfrak{a})$, and*

2. $S(R, \mathfrak{a}, A) := \bigcup_{S \in \text{Den}_l(R, \mathfrak{a}, A)} S$ is its largest element. In particular, $S_0(R) = S(R, 0, Q_l(R))$.
3. Let $S_i \in \text{Den}_l(R, \mathfrak{a}, A)$, $i \in I$, where I is an arbitrary non-empty set. Then $\langle S_i \mid i \in I \rangle \in \text{Den}_l(R, \mathfrak{a}, A)$ (see (13)). Moreover, $\langle S_i \mid i \in I \rangle$ is the least upper bound of the set $\{S_i\}_{i \in I}$ in $\text{Den}_l(R, \mathfrak{a}, A)$ and in $\text{Den}_l(R, \mathfrak{a})$.

Proof. 1. In view of Theorem 2.1, it suffices to show that if $S_1, S_2 \in \text{Den}_l(R, \mathfrak{a}, A)$ then $S_1 S_2 \in \text{Den}_l(R, \mathfrak{a}, A)$, i.e. $(S_1 S_2)^{-1} R = A$. By Theorem 2.1, $S_1 S_2 \in \text{Den}_l(R, \mathfrak{a})$. Notice that $A = S_1^{-1} R = S_2^{-1} R$. For each $s = s_1 \cdots s_n \in S_1 S_2$ where all $s_i \in S_1 \cup S_2$, and, for each $r \in R$,

$$s^{-1} r = s_n^{-1} \cdots s_1^{-1} r \in s_n^{-1} \cdots (s_2^{-1} A) \subseteq s_n^{-1} \cdots (s_3^{-1} A) \subseteq \cdots \subseteq A.$$

Therefore, $(S_1 S_2)^{-1} R = A$.

2. By Theorem 2.1, $S(R, \mathfrak{a}, A) = \langle S \mid S \in \text{Den}_l(R, \mathfrak{a}, A) \rangle \in \text{Den}_l(R, \mathfrak{a})$; and $S(R, \mathfrak{a}, A)^{-1} R = \text{inj lim } S^{-1} R = \text{inj lim } A = A$ where the injective limit is over $S \in \text{Den}_l(R, \mathfrak{a}, A)$.

3. Repeat the proof of statement 2 replacing $\text{Den}_l(R, \mathfrak{a}, A)$ by I . \square

Lemma 3.2 1. Let $S \in \text{Den}_l(R, \mathfrak{a})$, \mathfrak{b} be an ideal of the ring R such $\mathfrak{b} \subseteq \mathfrak{a}$, and $\pi : R \rightarrow R/\mathfrak{b}$, $a \mapsto \bar{a} = a + \mathfrak{b}$. Then $\pi(S) \in \text{Den}_l(R/\mathfrak{b}, \mathfrak{a}/\mathfrak{b})$ and $S^{-1} R \simeq \pi(S)^{-1} (R/\mathfrak{b})$.

2. Let $S_1, S_2 \in \text{Den}_l(R)$ and $S_1 \subseteq S_2$. Then

(a) $\mathfrak{a}_1 := \text{ass}(S_1) \subseteq \mathfrak{a}_2 := \text{ass}(S_2)$; there is the R -ring homomorphism $\varphi : S_1^{-1} R \rightarrow S_2^{-1} R$, $s^{-1} a \mapsto s^{-1} a$; and $\ker(\varphi) = S_1^{-1} (\mathfrak{a}_2/\mathfrak{a}_1)$.

(b) Let $\pi_1 : R \rightarrow R/\mathfrak{a}_1$, $a \mapsto \bar{a} = a + \mathfrak{a}_1$, and \tilde{S}_2 be the multiplicative submonoid of $(S_1^{-1} (R/\mathfrak{a}_1), \cdot)$ generated by $\pi_1(S_2)$ and $\pi_1(S_1)^{-1} = \{\bar{s}^{-1} \mid s \in S_1\}$. Then $\pi_1(S_2), \tilde{S}_2 \in \text{Den}_l(S_1^{-1} R, S_1^{-1} (\mathfrak{a}_2/\mathfrak{a}_1))$ and $\tilde{S}_2^{-1} (S_1^{-1} R) \simeq \pi_1(S_2)^{-1} (S_1^{-1} R) \simeq S_2^{-1} R$.

Proof. 1. Since π is an epimorphism and $\mathfrak{b} \subseteq \mathfrak{a}$, $\pi(S)$ is a left Ore set in R/\mathfrak{a} . Clearly, $\mathfrak{a}/\mathfrak{b} \subseteq \text{ass}(\pi(S))$. To prove that the inverse inclusion holds, let $\bar{a} \in \text{ass}(\pi(S))$. Then $0 = \bar{a}\bar{s}$ for some element $\bar{s} \in \pi(S)$. Then $b := sa \in \mathfrak{b}$. Since $\mathfrak{b} \subseteq \mathfrak{a}$, we can find an element $t \in S$ such that $tb = 0$. Then $(ts)a = 0$, and so $a \in \mathfrak{a}$. Therefore, $\text{ass}(\pi(S)) = \mathfrak{a}/\mathfrak{b}$. To prove that $\pi(S) \in \text{Den}_l(R/\mathfrak{b})$, it remains us to show that $\bar{a}\bar{s} = 0$, for some $\bar{a} \in R/\mathfrak{b}$ and $\bar{s} \in \pi(S)$, implies that $t\bar{a} = 0$ for some $t \in S$. Clearly, $b := as \in \mathfrak{b} \subseteq \mathfrak{a}$, and so $0 = s_1 b = (s_1 a)s$ for some $s_1 \in S$. It follows that $s_1 a \in \mathfrak{a}$ since $S \in \text{Den}_l(R, \mathfrak{a})$. We can find an element $s_2 \in S$ such that $s_2 s_1 a = 0$. Therefore, $\pi(S) \in \text{Den}_l(R/\mathfrak{b}, \mathfrak{a}/\mathfrak{b})$. It suffices to take $t = s_2 s_1 \in S$.

By the universal property of the ring $S^{-1} R$, there is the ring epimorphism

$$S^{-1} R \rightarrow \pi(S)^{-1} (R/\mathfrak{b}), \quad s^{-1} r \mapsto \pi(s)^{-1} \pi(r).$$

By the universal property of the ring $\pi(S)^{-1} (R/\mathfrak{b})$, there is the ring epimorphism $\pi(S)^{-1} (R/\mathfrak{b}) \rightarrow S^{-1} R$, $\pi(s)^{-1} \pi(r) \mapsto s^{-1} r$. Therefore, $S^{-1} R \simeq \pi(S)^{-1} (R/\mathfrak{b})$.

2a. The inclusion $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ is obvious. The image \bar{S}_1 of the Ore set S_1 under the ring epimorphism $\pi : R \rightarrow R/\mathfrak{a}_2$ is an Ore set in R/\mathfrak{a}_2 since $S_1 \subseteq S_2$ and $S_2 \in \text{Den}_l(R, \mathfrak{a}_2)$. Therefore, each element of the subring of $S_2^{-1} R$ generated by $\pi(R)$ and \bar{S}_1 has the form $\pi(s)^{-1} \pi(r)$ for some $s \in S_1$ and $r \in R$. By the universal property of the ring $S_1^{-1} R$, the map φ exists. An element $s^{-1} r \in S_1^{-1} R$, where $s \in S_1$ and $r \in R$, belongs to the kernel of φ iff $s^{-1} r = 0$ in $S_2^{-1} R$ iff $tr = 0$ for some $t \in S_2$ iff $r \in \mathfrak{a}_2$. Therefore, $\ker(\varphi) = S_1^{-1} (\mathfrak{a}_2/\mathfrak{a}_1)$.

2b. First, we prove that the set $\pi_1(S_2)$ (resp. \tilde{S}_2) is a left Ore set in $S_1^{-1} R$, i.e. we have to prove that for each element $\bar{s}_2 \in \pi_1(S_2)$ (resp. $\bar{s}_2 \in \tilde{S}_2$) and $s_1^{-1} r \in S_1^{-1} R$ where $s_1 \in S_1$ and $r \in R$, there exist elements $t \in \pi_1(S_2)$ (resp. $t \in \tilde{S}_2$) and $a \in S_1^{-1} R$ such that $ts_1^{-1} r = a\bar{s}_2$. In the ring $S_2^{-1} R$, the element $s_1^{-1} r s_2^{-1}$ can be written as $s_3^{-1} r_1$ for some $s_3 \in S_2$ and $r_1 \in R$. In $S_1^{-1} R$, $b := \bar{s}_3 s_1^{-1} r - r_1 \bar{s}_2 \in \ker(\varphi) = S_1^{-1} (\mathfrak{a}_2/\mathfrak{a}_1)$, by statement 2(a). Therefore, there exists an element $s_4 \in S_2$ such that $\bar{s}_4 b = 0$, and so $\bar{s}_4 \bar{s}_3 \cdot s_1^{-1} r = \bar{s}_4 r \cdot \bar{s}_2$. It suffices to take $t = \bar{s}_4 \bar{s}_3 \in \pi_1(S_2) \subseteq \tilde{S}_2$ and $a = \bar{s}_4 r$.

It is obvious that $S_1^{-1}(\mathfrak{a}_2/\mathfrak{a}_1) \subseteq \text{ass}(\pi_1(S_2)) \subseteq \text{ass}(\tilde{S}_2)$. If $u \in \text{ass}(\tilde{S}_2)$ then $su = 0$ for some $s \in \tilde{S}_2$. Then $0 = \varphi(su) = \varphi(s)\varphi(u)$, and so $u \in \ker(\varphi) = S_1^{-1}(\mathfrak{a}_2/\mathfrak{a}_1)$ (by statement 2(a)) since $\varphi(s)$ is a unit. Therefore, $S_1^{-1}(\mathfrak{a}_2/\mathfrak{a}_1) = \text{ass}(\pi_1(S_2)) = \text{ass}(\tilde{S}_2)$.

If $vs = 0$ for some elements $v \in S_1^{-1}R$ and $s \in \pi_1(S_2)$ (resp. $s \in \tilde{S}_2$). Then $0 = \varphi(v)\varphi(s)$, and so $\varphi(v) \in \ker(\varphi) = S_1^{-1}(\mathfrak{a}_2/\mathfrak{a}_1)$ since $\varphi(s)$ is a unit. This proves that $\pi_1(S_2) \in \text{Den}_l(S_1^{-1}R, S^{-1}(\mathfrak{a}_2/\mathfrak{a}_1))$ (resp. $\tilde{S}_2 \in \text{Den}_l(S_1^{-1}R, S^{-1}(\mathfrak{a}_2/\mathfrak{a}_1))$). Now, it is obvious that $\pi_1(S_2)^{-1}(S_1^{-1}R) \simeq S_2^{-1}R \simeq \tilde{S}_2^{-1}(S_1^{-1}R)$. \square

The set $(\text{Loc}_l(R, \mathfrak{a}), \rightarrow)$ is a poset where $A_1 \rightarrow A_2$ if $A_1 = S_1^{-1}R$ and $A_2 = S_2^{-1}R$ for some denominator sets $S_1, S_2 \in \text{Den}_l(R, \mathfrak{a})$ with $S_1 \subseteq S_2$, and $A_1 \rightarrow A_2$ is the inclusion map in Lemma 3.2.(2a). If (S'_1, S'_2) is another such a pair then, by Proposition 3.1.(1), $A_1 = S_1^{-1}R = S'^{-1}_1R = (S_1S'_1)^{-1}R \rightarrow A_2 = S_2^{-1}R = S'^{-1}_2R = (S_2S'_2)^{-1}R$; $S_1S'_1, S_2S'_2 \in \text{Den}_l(R, \mathfrak{a})$ with $S_1S'_1 \subseteq S_2S'_2$.

In the same way, the poset $(\text{Loc}_l(R), \rightarrow)$ is defined, i.e. $A_1 \rightarrow A_2$ if there exists $S_1, S_2 \in \text{Den}_l(R)$ such that $S_1 \subseteq S_2$, $A_1 = S_1^{-1}R$ and $A_2 = S_2^{-1}R$, $A_1 \rightarrow A_2$ stands for the map $\varphi : S_1^{-1}R \rightarrow S_2^{-1}R$ (Lemma 3.2.(2a)). The map

$$(\cdot)^{-1}R : \text{Den}_l(R) \rightarrow \text{Loc}_l(R), \quad S \mapsto S^{-1}R, \quad (2)$$

is an epimorphism of the posets $(\text{Den}_l(R), \subseteq)$ and $(\text{Loc}_l(R), \rightarrow)$. For each ideal $\mathfrak{a} \in \text{Ass}_l(R)$, it induces the epimorphism of posets $(\text{Den}_l(R, \mathfrak{a}), \subseteq)$ and $(\text{Loc}_l(R, \mathfrak{a}), \rightarrow)$,

$$(\cdot)^{-1}R : \text{Den}_l(R, \mathfrak{a}) \rightarrow \text{Loc}_l(R, \mathfrak{a}), \quad S \mapsto S^{-1}R. \quad (3)$$

The sets $\text{Den}_l(R)$ and $\text{Loc}_l(R)$ are the disjoint unions

$$\text{Den}_l(R) = \bigsqcup_{\mathfrak{a} \in \text{Ass}_l(R)} \text{Den}_l(R, \mathfrak{a}), \quad \text{Loc}_l(R) = \bigsqcup_{\mathfrak{a} \in \text{Ass}_l(R)} \text{Loc}_l(R, \mathfrak{a}). \quad (4)$$

For each ideal $\mathfrak{a} \in \text{Ass}_l(R)$, the set $\text{Den}_l(R, \mathfrak{a})$ is the disjoint union

$$\text{Den}_l(R, \mathfrak{a}) = \bigsqcup_{A \in \text{Loc}_l(R, \mathfrak{a})} \text{Den}_l(R, \mathfrak{a}, A). \quad (5)$$

Let $\text{LDen}_l(R, \mathfrak{a}) := \{S(R, \mathfrak{a}, A) \mid A \in \text{Loc}_l(R, \mathfrak{a})\}$, see Proposition 3.1.(2). The map

$$(\cdot)^{-1}R : \text{LDen}_l(R, \mathfrak{a}) \rightarrow \text{Loc}_l(R, \mathfrak{a}), \quad S \mapsto S^{-1}R, \quad (6)$$

is an isomorphism of posets.

For a left denominator set $S \in \text{Den}_l(R, \mathfrak{a})$, there are natural ring homomorphisms

$$R \rightarrow R/\mathfrak{a} \rightarrow S^{-1}R.$$

Lemma 3.3 and Proposition 3.4 establish connections between the left denominator sets $\text{Den}_l(R, \mathfrak{a})$, $\text{Den}_l(R/\mathfrak{a}, 0)$ and $\text{Den}_l(S^{-1}R, 0)$.

Let $S, T \in \text{Den}_l(R)$. The denominator set T is called *S-saturated* if $sr \in T$, for some $s \in S$ and $r \in R$, then $r \in T$, and if $r's' \in T$, for some $s' \in S$ and $r' \in R$, then $r' \in T$.

Lemma 3.3 *Let $S \in \text{Den}_l(R, \mathfrak{a})$, $\pi : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, and $\sigma : R \rightarrow S^{-1}R$, $r \mapsto r/1$.*

1. *Let $T \in \text{Den}_l(S^{-1}R, 0)$ be such that $\pi(S), \pi(S^{-1}) \subseteq T$. Then $T' := \sigma^{-1}(T) \in \text{Den}_l(R, \mathfrak{a})$, T' is S -saturated, $T = \{s^{-1}t' \mid s \in S, t' \in T'\}$, and $S^{-1}R \subseteq T'^{-1}R = T^{-1}R$.*
2. *$\pi^{-1}(S_0(R/\mathfrak{a})) = S_{\mathfrak{a}}(R)$, $\pi(S_{\mathfrak{a}}(R)) = S_0(R/\mathfrak{a})$ and $Q_{\mathfrak{a}}(R) = S_{\mathfrak{a}}(R)^{-1}R = Q_l(R/\mathfrak{a})$.*

Proof. 1. In the proof below we often identify the elements $\sigma(s)$, $s/1$ and s in order to avoid cumbersome notation. By the very definition, T' is a multiplicatively closed set that contains S . To prove that T' is a left Ore set in R we have to show that, for any $t' \in T'$ and $r \in R$, there exist elements $t'_1 \in T'$ and $a \in R$ such that $t'_1 r = at'$. Since $T \in \text{Den}_l(S^{-1}R, 0)$ and $\sigma(t') \in T$, $t\sigma(r) = s_1^{-1}r_1\sigma(t')$ for some elements $t \in T$, $s_1 \in S$ and $r_1 \in R$. The element t is equal to $s_2^{-1}r_2$ for some elements $s_2 \in S$ and $r_2 \in R$. Clearly, $r_2 \in T'$ since $\pi(S) \subseteq T$ and $r_2/1 = s_2 t \in T$. Fix an element $s_3 \in S$ such that $r_3/1 := s_3 s_1 s_2^{-1} \in \pi(R)$. Then $r_3 \in T'$ since $\pi(S)^{\pm 1} \subseteq T$. Multiplying the equality

$$s_1 s_2^{-1} r_2 \sigma(r) = r_1 \sigma(t')$$

by the element s_3 on the left we obtain the equality $r_3 r_2 \sigma(r) = s_3 r_1 \sigma(t') \in \pi(R)$. Hence $s_4 r_3 r_2 \cdot r = s_4 s_3 r_1 \cdot t'$ for some element $s_4 \in S$. Now, take $t'_1 := s_4 r_3 r_2 \in T'$ and $a := s_4 s_3 r_1 \in R$.

Since $S \subseteq T'$, $\mathfrak{a} \subseteq \text{ass}(T')$. If $\theta u = 0$ and $v\eta = 0$ for some elements $u, v \in R$ and $\theta, \eta \in T'$ then $0 = \sigma(\theta)\sigma(u)$ and $0 = \sigma(v)\sigma(\eta)$ and so $0 = \sigma(u) = \sigma(v)$ since the elements $\sigma(\theta)$ and $\sigma(\eta)$ are units. Then, $u, v \in \mathfrak{a}$. This proves that $T' \in \text{Den}_l(R, \mathfrak{a})$.

The denominator set T' is S -saturated since $sr = t' \in T'$ (resp. $rs = t' \in T'$) for some $s \in S$ and $r \in R$ implies $r/1 = s^{-1}t' \in T$ (resp. $r/1 = t's^{-1} \in T$), and so $r \in T'$. Since $\pi(S) \subseteq T$, $T = \{s^{-1}t' \mid s \in S, t' \in T'\}$. Then the inclusion and the equality, $S^{-1}R \subseteq T'^{-1}R = T^{-1}R$, are obvious.

2. Let $S' := \pi^{-1}(S_0(R/\mathfrak{a}))$. Since the map π is surjective we have $\pi(S') = S_0(R/\mathfrak{a})$. We aim to show that $S' = S_{\mathfrak{a}}(R)$.

Step 1: $S' \in \text{Ore}_l(R)$. We have to show that for any $s \in S'$ and $r \in R$ there exist elements $s_1 \in S'$ and $r_1 \in R$ such that $s_1 r = r_1 s$. Since $\pi(S') = S_0(R/\mathfrak{a}) \in \text{Ore}_l(R/\mathfrak{a})$, we have $\pi(s')\pi(r) = \pi(r')\pi(s)$ for some elements $s' \in S'$ and $r' \in R$. Then $\pi(s'r - r's) = 0$, and so $s'r - r's \in \mathfrak{a}$, hence $s''(s'r - r's) = 0$ for some $s'' \in S$. Note that $\pi(S) \in \text{Den}_l(R/\mathfrak{a}, 0)$ hence $\pi(S) \subseteq S_0(R/\mathfrak{a})$, and so $S \subseteq S'$. Now, $s_1 r = r_1 s$ where $s_1 = s''s' \in S'$ and $r_1 = s''r' \in R$. Therefore, $S' \in \text{Ore}_l(R)$.

Step 2: $\text{ass}(S') = \mathfrak{a}$. Since $S \subseteq S'$, we have the inclusion $\mathfrak{a} \subseteq \text{ass}(S')$. Let $r \in \text{ass}(S')$. Then $s'r = 0$ for some $s' \in S'$, and so $\pi(s')\pi(r) = 0$. It follows that $\pi(r) = 0$ since $\pi(s') \in S_0(R/\mathfrak{a})$, i.e. $r \in \ker(\pi) = \mathfrak{a}$. This means that $\text{ass}(S') = \mathfrak{a}$.

Step 3: $S' \in \text{Den}_l(R, \mathfrak{a})$. In view of Steps 1 and 2, we have to show that the equality $rs' = 0$ for some elements $r \in R$ and $s' \in S'$ implies that $s''r = 0$ for some $s'' \in S'$. Since $\pi(r)\pi(s') = 0$ and $\pi(s') \in S_0(R/\mathfrak{a}) = \pi(S')$ we have the equality $\pi(r) = 0$ and so $r \in \mathfrak{a}$. Then $s''r = 0$ for some element $s'' \in S \subseteq S'$.

Step 4: $S' = S_{\mathfrak{a}}(R)$. By Step 3, $S'^{-1}R = \pi(S')^{-1}R/\mathfrak{a} = S_0(R/\mathfrak{a})^{-1}R/\mathfrak{a} = Q_l(R/\mathfrak{a})$. Notice that $S_{\mathfrak{a}}(R)^{-1}R = \pi(S_{\mathfrak{a}}(R))^{-1}R/\mathfrak{a} \subseteq Q_l(R/\mathfrak{a})$. Since $\pi(S_{\mathfrak{a}}(R)) \in \text{Den}_l(R/\mathfrak{a}, 0)$, we have $\pi(S_{\mathfrak{a}}(R)) \subseteq S_0(R/\mathfrak{a})$, hence $S_{\mathfrak{a}}(R) \subseteq \pi^{-1}(S_0(R/\mathfrak{a})) = S' \subseteq S_{\mathfrak{a}}(R)$, i.e. $S' = S_{\mathfrak{a}}(R)$. \square

For $S_1, S_2 \in \text{Den}_l(R)$ such that $S_1 \subseteq S_2$, $[S_1, S_2] := \{T \in \text{Den}_l(R) \mid S_1 \subseteq T \subseteq S_2\}$ is an interval in the posed $\text{Den}_l(R)$. If, in addition, $S_1, S_2 \in \text{Den}_l(R, \mathfrak{a})$ then $[S_1, S_2] \subseteq \text{Den}_l(R, \mathfrak{a})$ since $S_1 \subseteq S \subseteq S_2$ implies $\mathfrak{a} = \text{ass}(S_1) \subseteq \text{ass}(S) \subseteq \text{ass}(S_2) = \mathfrak{a}$, i.e. $\text{ass}(S) = \mathfrak{a}$.

Proposition 3.4 *Let $S \in \text{Den}_l(R, \mathfrak{a})$; $\pi : R \rightarrow R/\mathfrak{a}$, $a \rightarrow \bar{a} = a + \mathfrak{a}$; $\sigma : R \rightarrow S^{-1}R$, $r \rightarrow r/1$; $G := \langle \pi(S), \pi(S)^{-1} \rangle \subseteq (S^{-1}R)^*$ (i.e. G is the subgroup of the group $(S^{-1}R)^*$ of units of the ring $S^{-1}R$ generated by $\pi(S)^{\pm 1}$).*

1. Let $[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}} := \{S_1 \in [\sigma^{-1}(G), S_{\mathfrak{a}}(R)] \mid \sigma^{-1}(G\pi(S_1)) = S_1\}$ and $[G, S_0(S^{-1}R)] := \{T \in \text{Den}_l(S^{-1}R, 0) \mid G \subseteq T \subseteq S_0(S^{-1}R)\}$. Then the map

$$[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}} \rightarrow [G, S_0(S^{-1}R)], \quad S_1 \mapsto \tilde{S}_1 := G\pi(S_1),$$

is an isomorphism of posets and abelian monoids with the inverse map $T \mapsto \sigma^{-1}(T)$ where $G\pi(S_1)$ is the multiplicative monoid generated by G and $\pi(S_1)$ in $S^{-1}R$. In particular,

$$G\pi(S_{\mathfrak{a}}(R)) = S_0(S^{-1}R), \quad S_{\mathfrak{a}}(R) = \sigma^{-1}(S_0(S^{-1}R)), \quad S_{\mathfrak{a}}(R)^{-1}R = Q_l(R/\mathfrak{a}),$$

the monoid $[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S-\text{com}}$ is a complete lattice (since $[G, S_0(S^{-1}R)]$ is a complete lattice, as an interval of the complete lattice $\text{Den}_l(S^{-1}R, 0)$, Corollary 2.5), and the map $S_1 \mapsto \widetilde{S}_1$ is a lattice isomorphism.

2. Consider the interval $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]$ in $\text{Den}_l(R/\mathfrak{a}, 0)$. Let $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}} := \{T \in [G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})] \mid GT \cap (R/\mathfrak{a}) = T\}$. Then $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}} \subseteq \text{Den}_l(S^{-1}R, 0)$ and the map

$$[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}} \rightarrow [G, S_0(S^{-1}R)], \quad T \mapsto GT,$$

is an isomorphism of posets and abelian monoids with the inverse map $T' \mapsto T' \cap (R/\mathfrak{a})$ where GT is the product in $\text{Den}_l(S^{-1}R, 0)$. In particular, the monoid $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}}$ is a complete lattice.

3. The map

$$[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S-\text{com}} \rightarrow [G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}}, \quad S_1 \mapsto G\pi(S_1) \cap (R/\mathfrak{a}),$$

is an isomorphism of posets and abelian monoids with the inverse map $S' \mapsto \pi^{-1}(S')$.

Proof. 1. The equality $\widetilde{\sigma^{-1}(G)} = G$ is obvious. Then, by Lemma 3.2.(2).(b), the map $\phi : S_1 \mapsto \widetilde{S}_1$ is well-defined. By Lemma 3.3.(1), the map $\psi : T \mapsto \sigma^{-1}(T)$ is well-defined and $\sigma^{-1}(T) = T$ (i.e. $\phi\psi = 1$) since $T = \{s^{-1}t \mid s \in S, t \in \sigma^{-1}(T)\}$ (Lemma 3.3.(1)) and $G \subseteq T$. By the very definition of the set $[S, S_{\mathfrak{a}}(R)]_{S-\text{com}}$, $\psi\phi = 1$, i.e. $\psi = \phi^{-1}$. For all elements $S_1, S_2 \in [S, S_{\mathfrak{a}}(R)]_{S-\text{com}}$,

$$\widetilde{S_1 \bigwedge S_2} = \widetilde{S_1} \widetilde{S_2} = G\pi(S_1)\pi(S_2) = G\pi(S_1)G\pi(S_2) = \widetilde{S_1} \widetilde{S_2} = \widetilde{S_1} \bigwedge \widetilde{S_2},$$

and so the map ϕ is an isomorphism of posets and abelian monoids. Therefore, $G\pi(S_{\mathfrak{a}}(R)) = S_0(S^{-1}R)$ and $S_{\mathfrak{a}}(R) = \sigma^{-1}(S_0(S^{-1}R))$. Then, $S_{\mathfrak{a}}(R)^{-1}R = Q_l(R/\mathfrak{a})$, by Lemma 3.3.(1). The rest is obvious.

2. Let $\phi : T \mapsto GT$ and $\psi : T' \mapsto T' \cap \overline{R}$ where $\overline{R} := R/\mathfrak{a}$.

Step 1: ϕ is well-defined. Since $G \in \text{Den}_l(S^{-1}R, 0)$ and $\phi(T) = GT$ (the product in $\text{Den}_l(S^{-1}R, 0)$), we have to show that $T \in \text{Den}_l(S^{-1}R, 0)$.

First, let us show that $T \in \text{Ore}_l(S^{-1}R)$, i.e. for each $s^{-1}r \in S^{-1}R$ (where $s \in S$ and $r \in \overline{R}$) and $t \in T$ we have to show that $t_1 s^{-1}r = at$ for some elements $t_1 \in T$ and $a \in S^{-1}R$. Since $T \in \text{Den}_l(\overline{R}, 0)$, $t'r = at$ for some $t' \in T$ and $a \in \overline{R}$. So, it suffices to take $t_1 = t's$ ($t_1 \in T$ since $\pi(S) \subseteq T$).

Let us show that $\text{ass}(T) = 0$ in $S^{-1}R$. Suppose that $t \cdot s^{-1}r = 0$ for some $t \in T$, $s \in S$ and $r \in R$, we have to show that $s^{-1}r = 0$. There exist elements $s_1 \in S$ and $t_2 \in \overline{R}$ such that $s_1 t = t_2 s$, hence $t_2 = s_1 t s^{-1} \in \overline{R} \cap GT = T$. Then $0 = s_1 \cdot 0 = s_1 t s^{-1} r = t_2 r$, hence $r = 0$ (since $T \in \text{Den}_l(\overline{R}, 0)$), and so $s^{-1}r = 0$, as required. Therefore, $\text{ass}(T) = 0$ in $S^{-1}R$. To finish the proof of Step 1, we have to prove that $s^{-1}rt = 0$ for some $s \in S$, $r \in R$ and $t \in T$ implies $s^{-1}r = 0$. This is obvious since $s^{-1}rt = 0$ implies $rt = 0$ in \overline{R} , hence $r = 0$ since $T \in \text{Den}_l(\overline{R}, 0)$, and so $s^{-1}r = 0$. Therefore, $T \in \text{Den}_l(S^{-1}R, 0)$ for all $T \in [G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}}$. In particular, $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}} \subseteq \text{Den}_l(S^{-1}R, 0)$.

Step 2: ψ is well-defined. Let $T' \in \text{Den}_l(S^{-1}R, 0)$. We have to show that $T := \overline{R} \cap T' \in \text{Den}_l(\overline{R}, 0)$. It is obvious that $\text{ass}_{\overline{R}}(T) = 0$ since $\text{ass}_{\overline{R}}(T) \subseteq \text{ass}_{S^{-1}R}(T') = 0$. If $rt = 0$ for some elements $r \in \overline{R}$ and $t \in T$ then $r = 0$ since $T \subseteq T'$, $T' \in \text{Den}_l(S^{-1}R, 0)$ and $\overline{R} \subseteq S^{-1}R$. It remains to show that $T \in \text{Ore}_l(\overline{R})$, that is, for any $r \in \overline{R}$ and $t \in T$, $t_1 r = r_1 t$ for some elements $r_1 \in \overline{R}$ and $t_1 \in T$. Since $T \subseteq T'$ and $T' \in \text{Den}_l(S^{-1}R, 0)$, $t'r = s_1^{-1}r'_1 t$ for some elements $t' \in T'$, $s_1 \in S$ and $r'_1 \in R$. Fix an element $s_2 \in S$ such that $t_1 := s_2 s_1 t' \in \overline{R}$. Then $t_1 \in \overline{R} \cap T' = T$ and $t_1 r = s_2 s_1 t' r = s_2 s_1 s_1^{-1} r'_1 t = s_2 r'_1 \cdot t = r_1 t$ where $r_1 := s_2 r'_1 \in \overline{R}$, as required.

By the very definition of the set $[G \cap \overline{R}, S_0(\overline{R})]$, $\psi\phi = 1$. To finish the proof of statement 2 it remains to show that $\phi\psi = 1$, i.e. $G(\overline{R} \cap T') = T'$ for all $T' \in [G, S_0(S^{-1}R)]$. Since $G, \overline{R} \cap T' \subseteq T'$, the inclusion $T'' := G(\overline{R} \cap T') \subseteq T'$ is obvious. The reverse inclusion follows from the fact that

any element $t' \in T'$ can be written as $s^{-1}t$ for some elements $s \in S \subseteq G$ and $t \in \overline{R}$, hence $t = st' \in \overline{R} \cap T' = T$.

3. Statement 3 follows from statements 1 and 2. \square

The elements of the set $[S, S_{\mathfrak{a}}(R)]_{S\text{-com}}$ are called *S-complete* and the elements of the set $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G\text{-com}}$ are called *G-complete*.

Lemma 3.5 *We keep the notation of Proposition 3.4.(1). If $S_1 \in [\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}}$ then S_1 is *S-saturated*.*

Proof. Notice that $\sigma^{-1}(G) \subseteq S_1 \subseteq S_{\mathfrak{a}}(R)$ and $S_1 = \sigma^{-1}(G\pi(S_1))$. If $s_1 := sr \in S_1$ (resp. $s_1 := rs \in S_1$) for some elements $s \in S$ and $r \in R$ then $r/1 = s^{-1}s_1 \in G\pi(S_1)$ (resp. $r/1 = s_1s^{-1} \in G\pi(S_1)$) hence $r \in \sigma^{-1}(G\pi(S_1)) = S_1$, i.e. S_1 is *S-saturated*. \square

The maximal left quotient rings of a ring.

Lemma 3.6 *Let $S_1 \subseteq S_2 \subseteq \dots \subseteq S_i \subseteq \dots$ be an ascending chain in $\text{Den}_l(R)$, $\mathfrak{a}_i := \text{ass}(S_i)$, $S := \bigcup_{i \geq 1} S_i$. Then $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots \subseteq \mathfrak{a}_i \subseteq \dots$ is the ascending chain in $\text{Ass}_l(R)$, $S \in \text{Den}_l(R, \mathfrak{a})$ where $\mathfrak{a} := \bigcup_{i \geq 1} \mathfrak{a}_i$, $S^{-1}R = \text{inj lim } S_i^{-1}R$ where $S_1^{-1}R \rightarrow S_2^{-1}R \rightarrow \dots \rightarrow S_i^{-1}R \rightarrow \dots$ (Lemma 3.2.(2a)).*

Proof. By Lemma 3.2.(2a), $S \in \text{Den}_l(R, \mathfrak{a})$. For each number $i = 1, 2, \dots$, define the ring homomorphism $\phi_i : S_i^{-1}R \rightarrow S^{-1}R$, $s^{-1}r \mapsto s^{-1}r$, and $\nu_i : S_i^{-1}R \rightarrow S_{i+1}^{-1}R$, $s^{-1}r \mapsto s^{-1}r$. Then $\phi_i = \phi_{i+1}\nu_i$ for all i . Hence, there is the ring homomorphism $\phi : \text{inj lim } S_i^{-1}R \rightarrow S^{-1}R$ which is a surjection since $S = \bigcup_{i \geq 1} S_i$ and has kernel 0 since $\mathfrak{a} = \bigcup_{i \geq 1} \mathfrak{a}_i$, i.e. ϕ is an isomorphism. \square

Consider the poset $(\text{Den}_l(R), \subseteq)$. For each element $S \in \text{Den}_l(R)$, let $[S, \cdot) := \{S' \in \text{Den}_l(R) \mid S \subseteq S'\}$.

Lemma 3.7 1. *For each element $S \in \text{Den}_l(R)$, there exists a maximal element in the poset $([S, \cdot), \subseteq)$.*

2. *The set $(\text{max.Den}_l(R), \subseteq)$ of maximal elements of the poset $(\text{Den}_l(R), \subseteq)$ is a non-empty set.*

Proof. 1. Statement 1 follows from Lemma 3.6 and Zorn's Lemma.

2. Statement 2 follows from statement 1 and the fact that the set $\text{max.Den}_l(R)$ is the set of maximal elements of the poset $(\{1\}, \cdot) = \text{Den}_l(R)$. \square

Definition. An element S of the set $\text{max.Den}_l(R)$ is called a *maximal left denominator set* of R and the ring $S^{-1}R$ is called a *maximal left quotient ring* of R . The intersection

$$\text{l.rad}(R) := \bigcap_{S \in \text{max.Den}_l(R)} \text{ass}(S)$$

is called the *left localization radical* of the ring R .

Proposition 3.8 *Let $\mathfrak{a} \in \text{Ass}_l(R)$, $Q := Q_{\mathfrak{a}}(R)$, Q^* be the group of units of the ring Q and $\sigma : R \rightarrow Q_{\mathfrak{a}}(R)$, $r \mapsto \frac{r}{1}$. Let $T \in \text{Den}_l(Q, \mathfrak{b})$ where $\mathfrak{b} \in \text{Ass}_l(Q)$ and Q^*T be the multiplicative sub-semigroup of (Q, \cdot) generated by Q^* and T . Then*

1. $Q^*T \in \text{Den}_l(Q, \mathfrak{b})$.

2. *If, in addition, $Q^* \subseteq T$ (eg, Q^*T from statement 1) then*

- (a) $T' := \sigma^{-1}(T) \in \text{Den}_l(R, \mathfrak{b}')$ where $\mathfrak{b}' := \sigma^{-1}(\mathfrak{b}) \supseteq \mathfrak{a}$, $S_{\mathfrak{a}}(R) \subseteq T'$, $T = Q^* \sigma(T')$ (i.e. the monoid T is generated by Q^* and $\sigma(T')$) and $T'^{-1}R = T^{-1}Q$ (i.e. the natural ring monomorphism $T'^{-1}R \rightarrow T^{-1}Q$, $t^{-1}r \mapsto t^{-1}r$, is an isomorphism).
- (b) $S_{\mathfrak{a}}(R) \subseteq S_{\mathfrak{b}'}(R)$ and $S_{\mathfrak{b}'}(R) = \sigma^{-1}(S_{\mathfrak{b}}(Q))$.
- (c) $Q_{\mathfrak{b}'}(R) = Q_l(Q/\mathfrak{b})$, i.e. the natural ring monomorphism $Q_{\mathfrak{b}'}(R) \rightarrow Q_l(Q/\mathfrak{b})$, $t^{-1}r \mapsto t^{-1}r$, is an isomorphism.

Proof. 1. Clearly, $\mathcal{T} := Q^*T$ is a multiplicative monoid. We have to show that $\mathcal{T} \in \text{Den}_l(Q, \mathfrak{b})$. An element s of \mathcal{T} is a product $s = q_1 t_1 q_2 t_2 \cdots q_n t_n$ for some elements $q_i \in Q$ and $t_i \in T$.

Step 1: $\mathcal{T} \in \text{Ore}_l(R)$. For any element $s \in \mathcal{T}$ and $a \in Q$, we have to find elements $s_1 \in \mathcal{T}$ and $a_1 \in Q$ such that $s_1 a = a_1 s$. We use induction on n . When $n = 1$, i.e. $s = q_1 t_1$, then $s_1 a = a' t_1$ for some elements $s_1 \in T$ and $a' \in Q$ since $T \in \text{Den}_l(Q, \mathfrak{b})$. It suffices to take $a_1 = a' q_1^{-1}$ since $s_1 a = (a' q_1^{-1}) \cdot q_1 t_1 = a_1 q_1 t_1$. Suppose that $n > 1$ and the statement is true for all $n' < n$. Then $s_1 a = a_n q_n t_n$ for some elements $s_n \in T$ and $a_n \in Q$. By induction, $s'_n a_n = a_1 q_1 t_1 \cdots q_{n-1} t_{n-1}$ for some elements $s'_n \in \mathcal{T}$ and $a_1 \in Q$. It suffices to take $s_1 = s'_n s_n$ since then

$$s_1 a = s'_n s_n a = s'_n a_n q_n t_n = a_1 q_1 t_1 \cdots q_{n-1} t_{n-1} q_n t_n = a_1 s.$$

Step 2: $\text{ass}(\mathcal{T}) = \mathfrak{b}$. We have to show that $sa = 0$ for some $a \in Q$ implies $a \in \mathfrak{b}$. We use induction on n . When $n = 1$, i.e. $s = q_1 t_1$ then $q_1 t_1 a = 0$ implies $t_1 a = 0$ (since q_1 is a unit), and so $a \in \mathfrak{b}$ since $T \in \text{Den}_l(Q, \mathfrak{b})$. Let $n > 1$. Suppose that the result is true for all $n' < n$. Then $0 = sa = q_1 t_1 \cdots q_{n-1} t_{n-1} \cdot (q_n t_n a)$ implies (by induction) $q_n t_n a \in \mathfrak{b}$, hence $t_n a \in \mathfrak{b}$ since q_n is a unit. Then $t_{n+1} t_n a = 0$ for some element $t_{n+1} \in T$ since $T \in \text{Den}_l(Q, \mathfrak{b})$, and finally $a \in \mathfrak{b}$ since $t_{n+1} t_n \in T$.

Step 3: $\mathcal{T} \in \text{Den}_l(Q, \mathfrak{b})$. We have to show that $as = 0$ for some $a \in Q$ implies $a \in \mathfrak{b}$. We use induction on n . When $n = 1$, i.e. $a q_1 t_1 = 0$, then $a q_1 \in \mathfrak{b}$ since $T \in \text{Den}_l(Q, \mathfrak{b})$, and so $a \in \mathfrak{b}$ since q_1 is a unit. Let $n > 1$. Suppose that the result is true for all $n' < n$. Then $0 = as = (a q_1 t_1) \cdot (q_2 t_2 \cdots q_n t_n)$ implies (by induction) $a_1 q_1 t_1 \in \mathfrak{b}$, hence $t_{n+1} a q_1 \in \mathfrak{b}$ for some element $t_{n+1} \in T$ (since $T \in \text{Den}_l(Q, \mathfrak{b})$), i.e. $t_{n+1} a \in \mathfrak{b}$ since q_1 is a unit. Therefore, there exists an element $t_{n+2} \in T$ such that $t_{n+2} t_{n+1} a = 0$. This means that $a \in \mathfrak{b}$ since $t_{n+2} t_{n+1} \in T \in \text{Den}_l(Q, \mathfrak{b})$.

2a. By the very definition, T' is a monoid and $\mathfrak{a} \subseteq \mathfrak{b}'$. By Lemma 3.3.(2), $Q_{\mathfrak{a}}(R) = Q_0(R/\mathfrak{a})$, $\sigma^{-1}(S_0(R/\mathfrak{a})) = S_{\mathfrak{a}}(R)$ and $\sigma(S_{\mathfrak{a}}(R)) = S_0(R/\mathfrak{a})$. Therefore, $S_{\mathfrak{a}}(R) \subseteq T'$ since $\sigma(S_{\mathfrak{a}}(R)) = S_0(R/\mathfrak{a}) \subseteq Q^* \subseteq T$. Since $Q^* \subseteq T$, we have the equality $T = Q^* \sigma(T')$. Let us show that $T' \in \text{Den}_l(R, \mathfrak{b}')$.

Step 1: $T' \in \text{Ore}_l(R)$. We have to show that for any elements $r \in R$ and $t' \in T'$ there exists elements $t'_1 \in T'$ and $r_1 \in R$ such that $t'_1 r = r_1 t'$. Since $T \in \text{Den}_l(Q, \mathfrak{b})$ and $\sigma(t') \in T$, we have the equality $t \sigma(r) = s^{-1} r_2 \sigma(t')$ in the ring Q for some elements $t \in T$, $s \in S_0(R/\mathfrak{a})$ and $r_2 \in R$. Since $S_0(R/\mathfrak{a}) \subseteq Q^*$ and $Q^* \subseteq T$ (by the assumption), the product $st \in T$. Fix an element $s_1 \in S_0(R/\mathfrak{a}) \subseteq Q^* \subseteq T$ such that $s_1 st = \sigma(t'_2)$ for some element $t'_2 \in R$, necessarily $t'_2 \in T'$. Then $s_1 = \sigma(s'_1)$ for some element $s'_1 \in S_{\mathfrak{a}}(R)$ since $\sigma(S_{\mathfrak{a}}(R)) = S_0(R/\mathfrak{a})$. By multiplying the equality $t \sigma(r) = s^{-1} r_2 \sigma(t')$ by the element $s_1 s$ on the left, we obtain the equality

$$\sigma(t'_2 r - s'_1 r_2 t') = 0,$$

i.e. $\alpha := t'_2 r - s'_1 r_2 t' \in \mathfrak{a}$. There exists an element $s_3 \in S_{\mathfrak{a}}(R) \subseteq T'$ such that $s_3 \alpha = 0$, i.e. $s_3 t'_2 \cdot r = s_3 s'_1 r_2 \cdot t'$. Now, it suffices to take $t'_1 := s_3 t'_2 \in T'$ and $r_1 := s_3 s'_1 r_2 \in R$.

Step 2: $\text{ass}(T') = \mathfrak{b}'$. Let $r \in \text{ass}(T')$, i.e. $t' r = 0$ for some $t' \in T'$. Then $\sigma(t') \sigma(r) = 0$ in Q , and so $\sigma(r) \in \mathfrak{b}$ since $\sigma(t') \in T$. Hence $r \in \mathfrak{b}'$. This means that $\text{ass}(T') = \mathfrak{b}'$.

Step 3: $T' \in \text{Den}_l(R, \mathfrak{b}')$. We have to show that if $rt' = 0$ for some $r \in R$ and $t' \in T'$ then $r \in \mathfrak{b}'$. Since $\sigma(r) \sigma(t') = 0$ and $\sigma(t') \in T$, we have the inclusion $\sigma(r) \in \mathfrak{b}$, hence $r \in \mathfrak{b}'$.

Since $\sigma(T') \subseteq T$ and $\text{ass}(T') = \mathfrak{b}'$, there is the (natural) ring monomorphism

$$T'^{-1}R \rightarrow T^{-1}R, \quad t'^{-1}r \mapsto t'^{-1}r,$$

where $t' \in T'$ and $r \in R$. The monomorphism is, in fact, an isomorphism since $T = Q^* \sigma(T')$, $S_{\mathfrak{a}}(R) \subseteq T'$ and $Q^* = \{\sigma(s)^{-1} \sigma(t) \mid \text{where } s, t \in S_{\mathfrak{a}}(R)\}$ (by Lemma 3.3.(2) and Theorem 2.8(1-3)).

2b. Let T and T' be as in statement 2a (it exists, by statement 1). By statement 2a, $S_{\mathfrak{a}}(R) \subseteq T'$ and $T' \in \text{Den}_l(R, \mathfrak{b}')$. Clearly, $T' \subseteq S_{\mathfrak{b}'}(R)$ since $S_{\mathfrak{b}'}(R)$ is the largest element of the poset $(\text{Den}_l(R, \mathfrak{b}'), \subseteq)$. Therefore, $S_{\mathfrak{a}}(R) \subseteq S_{\mathfrak{b}'}(R)$.

Let $T = S_{\mathfrak{b}}(Q)$. Then $Q^* \subseteq T$ by statement 1 and the maximality of $S_{\mathfrak{b}}(Q)$. By statement 2a, $T' := \sigma^{-1}(S_{\mathfrak{b}}(Q)) \subseteq S_{\mathfrak{b}'}(R)$. Consider the ring epimorphism $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$. Notice that $\mathfrak{a} \subseteq \mathfrak{b}'$. Applying Lemma 3.2.(2) in the situation $S_{\mathfrak{a}}(R) \subseteq S_{\mathfrak{b}'}(R)$, we see that $\pi_{\mathfrak{a}}(S_{\mathfrak{b}'}(R)) \in \text{Den}_l(Q, \mathfrak{b})$. Therefore, $S_{\mathfrak{b}'}(R) \subseteq T'$. This proves that $S_{\mathfrak{b}'}(R) = T'$.

2c. Applying statement 2a in the situation $T = S_{\mathfrak{b}}(Q) \supseteq Q^*$ and using the fact that $S_{\mathfrak{b}'}(R) := \sigma^{-1}(S_{\mathfrak{b}}(Q)) = \sigma^{-1}(T)$, we see that $S_{\mathfrak{b}'}(R)^{-1}R = S_{\mathfrak{b}}(Q)^{-1}Q$, i.e. $Q_{\mathfrak{b}'}(R) = Q_l(Q/\mathfrak{b})$ since $Q_l(Q/\mathfrak{b}) = S_{\mathfrak{b}}(Q)^{-1}Q$ (by Lemma 3.3.(2)). \square

Let $\text{max.Ass}_l(R)$ be the set of maximal elements of the poset $(\text{Ass}_l(R), \subseteq)$. It is a subset of the set

$$\text{ass.max.Den}_l(R) := \{\text{ass}(S) \mid S \in \text{max.Den}_l(R)\} \quad (7)$$

which is a non-empty set, by Lemma 3.7.(2). Let $\text{max.Loc}_l(R)$ be the set of maximal elements of the poset $(\text{Loc}_l(R), \rightarrow)$. By the very definition of $\text{Loc}_l(R)$ and by Lemma 3.3.(2),

$$\text{max.Loc}_l(R) = \{S^{-1}R \mid S \in \text{max.Den}_l(R)\} = \{Q_l(R/\mathfrak{a}) \mid \mathfrak{a} \in \text{ass.max.Den}_l(R)\}. \quad (8)$$

Definition. Each element of $\text{max.Loc}_l(R)$ is called a *maximal left localization ring* (or a *maximal left quotient ring*) of the ring R .

Theorem 3.9 *Let $S \in \text{max.Den}_l(R)$, $A = S^{-1}R$, A^* be the group of units of the ring A ; $\mathfrak{a} := \text{ass}(S)$, $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, and $\sigma_{\mathfrak{a}} : R \rightarrow A$, $r \mapsto \frac{r}{1}$. Then*

1. $S = S_{\mathfrak{a}}(R)$, $S = \pi_{\mathfrak{a}}^{-1}(S_0(R/\mathfrak{a}))$, $\pi_{\mathfrak{a}}(S) = S_0(R/\mathfrak{a})$ and $A = S_0(R/\mathfrak{a})^{-1}R/\mathfrak{a} = Q_l(R/\mathfrak{a})$.
2. $S_0(A) = A^*$ and $S_0(A) \cap (R/\mathfrak{a}) = S_0(R/\mathfrak{a})$.
3. $S = \sigma_{\mathfrak{a}}^{-1}(A^*)$.
4. $A^* = \langle \pi_{\mathfrak{a}}(S), \pi_{\mathfrak{a}}(S)^{-1} \rangle$, i.e. the group of units of the ring A is generated by the sets $\pi_{\mathfrak{a}}(S)$ and $\pi_{\mathfrak{a}}^{-1}(S) := \{\pi_{\mathfrak{a}}(s)^{-1} \mid s \in S\}$.
5. $A^* = \{\pi_{\mathfrak{a}}(s)^{-1} \pi_{\mathfrak{a}}(t) \mid s, t \in S\}$.
6. $Q_l(A) = A$ and $\text{Ass}_l(A) = \{0\}$. In particular, if $T \in \text{Den}_l(A, 0)$ then $T \subseteq A^*$.

Proof. 1. Since $S = S_{\mathfrak{a}}(R)$ (by Theorem 2.1.(2)), statement 1 follows from Lemma 3.3.(2).

2. $S_0(A) \stackrel{\text{st.1}}{=} S_0(Q_l(R/\mathfrak{a})) \stackrel{\text{Thm 2.8.(1)}}{=} Q_l(R/\mathfrak{a})^* \stackrel{\text{st.1}}{=} A^*$ and $S_0(A) \cap (R/\mathfrak{a}) \stackrel{\text{st.1}}{=} S_0(Q_l(R/\mathfrak{a})) \cap (R/\mathfrak{a}) \stackrel{\text{Thm 2.8.(1)}}{=} S_0(R/\mathfrak{a})$.

3. $S = S_{\mathfrak{a}}(R) \stackrel{\text{Pr 3.4.(1)}}{=} \sigma_{\mathfrak{a}}^{-1}(S_0(S_{\mathfrak{a}}(R)^{-1}R)) = \sigma_{\mathfrak{a}}^{-1}(S_0(Q_l(R/\mathfrak{a}))) \stackrel{\text{Thm 2.8.(1)}}{=} \sigma_{\mathfrak{a}}^{-1}(Q_l(R/\mathfrak{a})^*) \stackrel{\text{st.1}}{=} \sigma_{\mathfrak{a}}^{-1}(A^*)$.

4. $A^* \stackrel{\text{st.1}}{=} Q_l(R/\mathfrak{a})^* \stackrel{\text{Thm 2.8.(3)}}{=} \langle S_0(R/\mathfrak{a}), S_0(R/\mathfrak{a})^{-1} \rangle \stackrel{\text{st.1}}{=} \langle \pi_{\mathfrak{a}}(S), \pi_{\mathfrak{a}}(S)^{-1} \rangle$.

5. $A^* \stackrel{\text{st.1}}{=} Q_l(R/\mathfrak{a})^* \stackrel{\text{Thm 2.8.(3)}}{=} \{p^{-1}q \mid p, q \in S_0(R/\mathfrak{a})\} = \{\pi_{\mathfrak{a}}(s)^{-1} \pi_{\mathfrak{a}}(t) \mid s, t \in S\}$ since $\pi_{\mathfrak{a}}(S) = S_0(R/\mathfrak{a})$, by statement 1.

6. $Q_l(A) \stackrel{\text{st.1}}{=} Q_l(Q_l(R/\mathfrak{a})) \stackrel{\text{Thm 2.8.(4)}}{=} Q_l(R/\mathfrak{a}) \stackrel{\text{st.1}}{=} A$. Since $S_0(A) \stackrel{\text{st.1}}{=} S_0(Q_l(R/\mathfrak{a})) \stackrel{\text{Thm 2.8.(1)}}{=} Q_l(R/\mathfrak{a})^* = A^*$, $T \subseteq A^*$. The fact that $\text{Ass}_l(A) = \{0\}$ follows from Proposition 3.8.(2) and the maximality of A . \square

The next theorem is a criterion of when a ring $A \in \text{Loc}_l(R, \mathfrak{a})$ is equal to $Q_{\mathfrak{a}}(R)$.

Theorem 3.10 *Let $A \in \text{Loc}_l(R, \mathfrak{a})$, i.e. $A = S^{-1}R$ for some $S \in \text{Den}_l(R, \mathfrak{a})$. Then $A = Q_{\mathfrak{a}}(R)$ iff $Q_l(A) = A$.*

Proof. If $A = Q_{\mathfrak{a}}(R)$ then $A = Q_l(R/\mathfrak{a})$, by Lemma 3.3.(2), and so $Q_l(A) = Q_l(Q_l(R/\mathfrak{a})) = Q_l(R/\mathfrak{a}) = A$, by Theorem 2.8.(4).

If $A \neq Q_{\mathfrak{a}}(R)$ then the monomorphism $A \rightarrow Q_{\mathfrak{a}}(R)$, $a \mapsto a$, is not surjective. Let $S_1 := S$ and $S_2 := S_{\mathfrak{a}}(R)$. Then $S_1 \subseteq S_2$ and, by Lemma 3.2.(2), $Q_{\mathfrak{a}}(R) \simeq \pi_1(S_2)^{-1}(S_1^{-1}R) \simeq \pi_1(S_2)^{-1}A$ where $\pi_1 : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$. Therefore, $A \neq Q_l(A)$. \square

Left localization maximal rings. We introduce a new class of rings, the left localization maximal rings, which turn out to be precisely the class of maximal left quotient rings of all rings. As a result, we have a characterization of the maximal left quotient rings of a ring (Theorem 3.11).

Definition. A ring A is called a *left localization maximal ring* if $A = Q_l(A)$ and $\text{Ass}_l(A) = \{0\}$. A ring A is called a *right localization maximal ring* if $A = Q_r(A)$ and $\text{Ass}_r(A) = \{0\}$. A ring A which is a left and right localization maximal ring is called a *left and right localization maximal ring* (i.e. $Q_l(A) = A = Q_r(A)$ and $\text{Ass}_l(A) = \text{Ass}_r(A) = \{0\}$).

Example. Let A be a simple ring. Then $Q_l(A)$ is a left localization maximal ring and $Q_r(A)$ is a right localization maximal ring.

Example. A division ring is a (left and right) localization maximal ring. More generally, a simple artinian algebra (i.e. the matrix algebra over a division ring) is a (left and right) localization maximal ring.

The next theorem is a criterion of when a left quotient ring of a ring is a maximal left quotient ring of the ring.

Theorem 3.11 *Let a ring A be a left localization of a ring R , i.e. $A \in \text{Loc}_l(R, \mathfrak{a})$ for some $\mathfrak{a} \in \text{Ass}_l(R)$. Then $A \in \text{max.Loc}_l(R)$ iff $Q_l(A) = A$ and $\text{Ass}_l(A) = \{0\}$, i.e. A is a left localization maximal ring.*

Proof. (\Rightarrow) Theorem 3.9.(6).
 (\Leftarrow) Proposition 3.8. \square

The next corollary is a criterion of when $S_{\mathfrak{a}_1}(R) \subseteq S_{\mathfrak{a}_2}(R)$ where $\mathfrak{a}_1, \mathfrak{a}_2 \in \text{Ass}_l(R)$.

Corollary 3.12 *Let $\mathfrak{a}_1, \mathfrak{a}_2 \in \text{Ass}_l(R)$ and $\sigma_i : R \rightarrow Q_{\mathfrak{a}_i}(R)$, $r \mapsto r/1$, for $i = 1, 2$. Then $S_{\mathfrak{a}_1}(R) \subseteq S_{\mathfrak{a}_2}(R)$ iff $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and $\sigma_2(S_{\mathfrak{a}_1}(R)) \subseteq Q_{\mathfrak{a}_2}(R)^*$.*

Proof. (\Rightarrow) By Lemma 3.2.(2a), $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and, by Lemma 3.2.(2b), $\sigma_2(S_{\mathfrak{a}_1}(R)) \subseteq Q_{\mathfrak{a}_2}(R)^*$.

(\Leftarrow) Let $S_i := S_{\mathfrak{a}_i}(R)$ and $Q_i := Q_{\mathfrak{a}_i}(R)$ for $i = 1, 2$. Let Q' be the subring of Q_2 generated by R/\mathfrak{a}_2 and $\sigma_2(S_1)^{\pm 1}$. Since $S_1 \in \text{Den}_l(R, \mathfrak{a}_1)$, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and $\sigma_2(S_1) \subseteq Q_2^*$, every element of Q' has the form $\sigma_2(s_1)^{-1}\sigma_2(r)$ for some elements $s_1 \in S_1$ and $r \in R$. By the universal property of $Q_1 = S_1^{-1}R$, there exists a ring homomorphism $Q_1 \rightarrow Q_2$ and so we have the commutative diagram of ring homomorphisms:

$$\begin{array}{ccccc} R & \longrightarrow & R/\mathfrak{a}_1 & \longrightarrow & Q_1 \\ \downarrow = & & \downarrow & & \downarrow \\ R & \longrightarrow & R/\mathfrak{a}_2 & \longrightarrow & Q_2. \end{array}$$

Since $S_i = \sigma_i^{-1}(Q_i^*)$ for $i = 1, 2$ (Lemma 3.3.(2) and Theorem 2.8), using the commutative diagram we have the inclusion $S_1 \subseteq S_2$. \square

Theorem 3.13 Let $S \in \max.\text{Den}_l(R)$, $A = S^{-1}R$, $\mathfrak{a} = \text{ass}(S)$ and $\sigma_{\mathfrak{a}} : R \rightarrow A$, $r \mapsto \frac{r}{1}$. Then the following statements are equivalent.

1. A is a semi-simple ring.
2. $Q_{cl}(R/\mathfrak{a})$ exists and is a semi-simple ring.

If one of these conditions holds then $A = Q_{cl}(R)$ and $S = \sigma_{\mathfrak{a}}^{-1}(Q_{cl}(R)^*)$.

Proof. 1. Since $A = Q_l(R/\mathfrak{a})$ and $S = \sigma_{\mathfrak{a}}^{-1}(A^*)$, by Theorem 3.9.(1,3), the results follow from Theorem 2.9. \square

The ideal $\mathfrak{p}(S)$ and the set $\mathcal{J}_l(R, S)$.

Proposition 3.14 Let $S_1, S_2 \in \text{Ore}_l(R)$ be such that $\mathfrak{a}_1 := \text{ass}(S_1) \subseteq \mathfrak{a}_2 := \text{ass}(S_2)$. Then

1. $S_1 S_2 \in \text{Ore}_l(R, \mathfrak{a})$ such that $\mathfrak{a}_2 \subseteq \mathfrak{a} := \text{ass}(S_1 S_2)$ where $S_1 S_2 := \langle S_1, S_2 \rangle$ is the sub-semigroup of (R, \cdot) generated by S_1 and S_2 .
2. If, in addition, $S_1, S_2, S \in \text{Den}_l(R)$ then for each $i = 1, 2$ there is the ring homomorphism $S_i^{-1}R \rightarrow S^{-1}R$, $s^{-1}r \mapsto s^{-1}r$, with kernel $S_i^{-1}\mathfrak{a}$.

Proof. 1. *Step 1:* $0 \notin S := \langle S_1, S_2 \rangle$. Suppose that $0 \in S$, we seek a contradiction. Then $s_1 s_2 \cdots s_n = 0$ for some elements $s_i \in S_1 \cup S_2$ and $n \geq 1$. Clearly, $n \geq 2$. We may assume that n is the least possible. Then, by the minimality of n , either all $s_{\text{even}} \in S_1$ and $s_{\text{odd}} \in S_2$ or otherwise $s_{\text{even}} \in S_2$ and $s_{\text{odd}} \in S_1$. If $s_1 \in S_1$ then $s_2 \in S_2$ and $s_2 s_3 \cdots s_n \in \mathfrak{a}_1 \subseteq \mathfrak{a}_2$, hence $(s'_2 s_2) s_3 \cdots s_n = 0$ for some element $s'_2 \in S_2$. This contradicts to the minimality of n since $s'_2 s_2 \in S_2$. So, $s_1 \in S_2$. If $s_n \in S_1$ then $s'_1 s_1 s_2 \cdots s_n = 0$ for some element $s'_1 \in S_1$. This is not possible by the previous case. Therefore $s_1, s_n \in S_2$. Then $(s'_n s_1) s_2 \cdots s_{n-1} = 0$ for some element $s'_n \in S_2$. This contradicts to the minimality of n since $s'_n s_1 \in S_2$. Therefore, $0 \notin S$.

Step 2: $S \in \text{Ore}_l(R)$. We have to show that for any elements $s \in S$ and $r \in R$ there exist elements $s' \in S$ and $r' \in R$ such that $s'r = r's$. To prove this we use induction on n where $s = s_1 s_2 \cdots s_n$, $s_i \in S_1 \cup S_2$. The result is obvious when $n = 1$ since $S_1, S_2 \in \text{Ore}_l(R)$. Suppose that $n > 1$ and the result is true for all $n' < n$. Fix $t_1 \in S_1 \cup S_2$ such that $t_1 r = r_1 s_n$ for some element $r_1 \in R$. By induction, $t_2 r_1 = r' s_1 \cdots s_{n-1}$ for some $t_2 \in S$ and $r' \in R$. Then $t_2 t_1 r = t_2 r_1 s_n = r' s_1 \cdots s_n$. It suffices to take $s' = t_2 t_1$. By Step 2, \mathfrak{a} is an ideal of the ring R such that $\mathfrak{a} \neq R$, by Step 1. Clearly, $\mathfrak{a}_2 \subseteq \mathfrak{a}$.

2. Statement 2 follows from statement 1 and Lemma 3.2.(2a). \square

In order to give an answer to the question of when an Ore set S of a ring is a denominator set or, more generally, when the image of S is a denominator set in a factor ring, we introduce the ideal $\mathfrak{p}(S)$.

Let \mathcal{W} be the family of all *ordinal* numbers. For each left Ore set $S \in \text{Ore}_l(R)$, we attach the ideal of the ring R ,

$$\mathfrak{p}(S) := \mathfrak{p}(S)_l := \bigcup_{\alpha \in \mathcal{W}} \mathfrak{p}_{\alpha} \quad (9)$$

where the ideals \mathfrak{p}_{α} of the ring R are defined recursively as follows: $\mathfrak{p}_1 := \text{ass}_l(S, R) + \text{ass}_r(S, R)R$ where $\text{ass}_l(S, R) := \{r \in R \mid sr = 0 \text{ for some } s = s(r) \in S\}$ and $\text{ass}_r(S, R) := \{r \in R \mid rs = 0 \text{ for some } s = s(r) \in S\}$. Note that $\text{ass}_l(S, R)$ is an ideal of the ring R since $S \in \text{Ore}_l(R)$ and $\text{ass}_r(S, R)$ is a right ideal of the ring R .

$$\mathfrak{p}_{\alpha+1} := \pi_{\alpha}^{-1}(\text{ass}_l(\pi_{\alpha}(S), R/\mathfrak{p}_{\alpha}) + \text{ass}_r(\pi_{\alpha}(S), R/\mathfrak{p}_{\alpha}) \cdot R/\mathfrak{p}_{\alpha}) \quad (10)$$

where $\pi_{\alpha} : R \rightarrow R/\mathfrak{p}_{\alpha}$, $a \mapsto a + \mathfrak{p}_{\alpha}$. If α is a limit ordinal then

$$\mathfrak{p}_{\alpha} = \bigcup_{\beta < \alpha} \mathfrak{p}_{\beta}. \quad (11)$$

By the very definition, if $\alpha \leq \beta$ then $\mathfrak{p}_\alpha \leq \mathfrak{p}_\beta$. In a similar fashion, for a *right* Ore set $S \in \text{Ore}_r(R)$, we can define the ideal $\mathfrak{p}(S)_r$. For each left Ore set $S \in \text{Ore}_l(R)$, let

$$\mathcal{J}_l(R, S) := \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ such that } \pi_{\mathfrak{a}}(S) \in \text{Den}_l(R/\mathfrak{a}, 0)\}$$

where $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$.

Theorem 3.15 *Let $S \in \text{Ore}_l(R)$ and $\mathfrak{a} = \text{ass}(S)$. Then*

1. $\mathcal{J}_l(R, S) \neq \emptyset$ iff $\mathfrak{p}(S) \neq R$.
2. If $\mathcal{J}_l(R, S) \neq \emptyset$ then $\mathfrak{p}(S)$ is the least (with respect to inclusion) element of $\mathcal{J}_l(R, S)$.
3. If, in addition, $S \in \text{Den}_l(R, \mathfrak{a})$ then $\mathfrak{p}(S) = \mathfrak{a}$.
4. If the ring R satisfies the ascending chain condition on annihilator right ideals then $S \in \text{Den}_l(R, \mathfrak{a})$ and $\mathfrak{p}(S) = \mathfrak{a}$.

Proof. 1. (\Rightarrow) Suppose that $\mathcal{J}_l(R, S) \neq \emptyset$. Fix $\mathfrak{a} \in \mathcal{J}_l(R, S)$. Clearly, $\text{ass}_l(S, R), \text{ass}_r(S, R) \subseteq \mathfrak{a}$, hence $\mathfrak{p}_1 \subseteq \mathfrak{a}$. If $\mathfrak{p}_\alpha \subseteq \mathfrak{a}$ then $\mathfrak{p}_{\alpha+1} \subseteq \mathfrak{a}$ since $sa, bt \in \mathfrak{p}_\alpha \subseteq \mathfrak{a}$ for some $s, t \in S$ and $a, b \in R$ implies $a, b \in \mathfrak{a}$. If α is a limit ordinal and $\mathfrak{p}_\beta \subseteq \mathfrak{a}$ for all $\beta < \alpha$ then $\mathfrak{p}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{p}_\beta \subseteq \mathfrak{a}$. Therefore, $\mathfrak{p}(S) \subseteq \mathfrak{a}$, and so $\mathfrak{p}(S) \neq R$.

(\Leftarrow) Suppose that $\mathfrak{p} := \mathfrak{p}(S) \neq R$. We claim that $\pi_{\mathfrak{p}}(S) \in \text{Den}_l(R/\mathfrak{p}, 0)$ where $\pi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$, $a \mapsto a + \mathfrak{p}$. If $\pi_{\mathfrak{p}}(s)\pi_{\mathfrak{p}}(a) = 0$ and $\pi_{\mathfrak{p}}(b)\pi_{\mathfrak{p}}(t) = 0$ in R/\mathfrak{p} for some elements $s, t \in S$ and $a, b \in R$ then $sa, bt \in \mathfrak{p} = \bigcup_{\alpha \in \mathbb{W}} \mathfrak{p}_\alpha$, i.e. $sa, bt \in \mathfrak{p}_\alpha$ for some ordinal number α . Therefore, $a, b \in \mathfrak{p}_{\alpha+1} \subseteq \mathfrak{p}$, i.e. $\pi_{\mathfrak{p}}(a) = \pi_{\mathfrak{p}}(b) = 0$. This proves that $\text{ass}_l(\pi_{\mathfrak{p}}(S), R/\mathfrak{p}) = 0$ and $\text{ass}_r(\pi_{\mathfrak{p}}(S), R/\mathfrak{p}) = 0$. To finish the proof of the fact that $\pi_{\mathfrak{p}}(S) \in \text{Den}_l(R/\mathfrak{p}, 0)$ we have to show that $\pi_{\mathfrak{p}}(S) \in \text{Ore}_l(R/\mathfrak{p})$, but this is obvious as epimorphisms respect left Ore sets provided their images are multiplicatively closed sets which is the case for $\pi_{\mathfrak{p}}(S)$ as $\mathfrak{p}(S) \neq R$.

2. In statement 1 we proved that $\mathfrak{p}(S) \subseteq \mathfrak{a}$ for all ideals $\mathfrak{a} \in \mathcal{J}_l(R, S)$ and $\mathfrak{p}(S) \in \mathcal{J}_l(R, S)$, i.e. $\mathfrak{p}(S)$ is the least element of the poset $(\mathcal{J}_l(R, S), \subseteq)$.

3. Statement 3 is obvious.

4. By Lemma 1.1.2, [3], $S \in \text{Den}_l(R, \mathfrak{a})$ and so $\mathfrak{p}(S) = \mathfrak{a}$, by statement 3. \square

Localizable left Ore sets, a slight generalization of Ore's method of localization.

Definition. A left Ore set $S \in \text{Ore}_l(R)$ is called *localizable* if there exists a ring homomorphism $\varphi : R \rightarrow R'$ where R' is a ring such that, for all $s \in S$, $\varphi(s)$ is a unit of the ring R' . The set of all localizable left Ore sets in R is denoted by $\text{ore}_l(R)$. By Theorem 3.15,

$$\text{ore}_l(R) = \{S \in \text{Ore}_l(R) \mid \mathfrak{p}(S) \neq R\}. \quad (12)$$

Corollary 3.16 *Let S be a localizable left Ore set in a ring R . Then there exists an ordered pair (Q, f) where Q is a ring and $f : R \rightarrow Q$ is a ring homomorphism such that*

(i) *for all $s \in S$, $f(s)$ is a unit in Q ;*

and if (Q', f') is another pair satisfying the condition (i) then there is a unique ring homomorphism $h : Q \rightarrow Q'$ such that $f' = hf$. The ring Q is unique up to isomorphism. The ring Q is isomorphic to the left localization of the ring $R/\mathfrak{p}(S)$ at the left denominator set $\pi(S) \in \text{Den}_l(R/\mathfrak{p}(S), 0)$ where the ideal $\mathfrak{p}(S)$ of the ring R is defined in (11) and $\pi : R \rightarrow R/\mathfrak{p}(S)$, $a \mapsto a + \mathfrak{p}(S)$.

Proof. Recall that $\mathfrak{p}(S) = \bigcup_{\alpha \in \mathbb{W}} \mathfrak{p}_\alpha$. By induction on α , all $\mathfrak{p}_\alpha \subseteq \ker(f')$. Therefore, $\mathfrak{p}(S) \subseteq \ker(f')$. Notice that $Q' \simeq \pi_{\mathfrak{a}}(S)^{-1}(R/\mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{J}_l(R, S)$. Let us identify this rings via the isomorphism above, then $f'(R) = R/\mathfrak{a}$. By Theorem 3.15.(2), $\mathfrak{p}(S) \subseteq \mathfrak{a}$, and so there is a natural ring homomorphism $\varphi : R/\mathfrak{p}(S) \rightarrow \pi_{\mathfrak{a}}(S)^{-1}(R/\mathfrak{a})$ such that, for all $s \in S$, $\varphi(s)$ is a unit. By the universal property of the ring $\pi_{\mathfrak{p}(S)}(S)^{-1}(R/\mathfrak{p}(S))$ there is a *unique* homomorphism $h : \pi_{\mathfrak{p}(S)}(S)^{-1}(R/\mathfrak{p}(S)) \rightarrow Q'$ such that $f' = hf$. This proves existence of the ring Q . It is unique up to isomorphism as it follows at once from the uniqueness of h . \square

The set $\text{LDen}_l(R) := \{S_{\mathfrak{a}}(R) \mid \mathfrak{a} \in \text{Ass}_l(R)\}$ is the set of largest left denominator sets of the ring R . Let $\min.\text{LDen}_l(R, \mathfrak{a})$ and $\min.\text{Loc}_l(R, \mathfrak{a})$ be the sets of minimal elements of the posets $\text{LDen}_l(R, \mathfrak{a})$ and $\text{Loc}_l(R, \mathfrak{a})$ respectively.

4 The largest quotient ring of a ring

In this section, the two-sided analogues of some results of Sections 2 and 3 are proved. The proofs are dropped in all cases where the two-sided analogues are direct corollaries of the one-sided results. In particular, it is proved existence of the largest quotient ring of a ring (Theorem 4.1) and that all Ore sets are localizable (Theorem 4.15). The results of this section are used in Section 5.

Notation:

- $\text{Ore}(R) := \text{Ore}_l(R) \cap \text{Ore}_r(R) = \{S \mid S \text{ is a left and right Ore set in } R\};$
- $\text{Den}(R) := \text{Den}_l(R) \cap \text{Den}_r(R) = \{S \mid S \text{ is a left and right denominator set in } R\}$. For each $S \in \text{Den}(R)$, $\text{ass}(S) := \text{ass}_l(S) = \text{ass}_r(S)$ is an ideal of the ring R where $\text{ass}_l(S) := \{r \in R \mid sr = 0 \text{ for some } s = s(r) \in S\}$ and $\text{ass}_r(S) := \{r \in R \mid rs = 0 \text{ for some } s = s(r) \in S\};$
- $\text{Loc}(R) := \{S^{-1}R = RS^{-1} \mid S \in \text{Den}(R)\};$
- $\text{Ass}(R) := \{\text{ass}(S) \mid S \in \text{Den}(R)\};$
- $\text{Den}(R, \mathfrak{a}) := \{S \in \text{Den}(R) \mid \text{ass}(S) = \mathfrak{a}\}$ where $\mathfrak{a} \in \text{Ass}(R);$
- $S_{\mathfrak{a}} = S_{\mathfrak{a}}(R)$ is the *largest element* of the poset $(\text{Den}(R, \mathfrak{a}), \subseteq)$ and $Q_{\mathfrak{a}}(R) := S_{\mathfrak{a}}^{-1}R = RS_{\mathfrak{a}}^{-1}$ is the *largest (two-sided) quotient ring associated to \mathfrak{a}* , $S_{\mathfrak{a}}$ exists (Theorem 4.1.(2));
- In particular, $S_0 = S_0(R)$ is the largest element of the poset $(\text{Den}(R, 0), \subseteq)$ and $Q(R) := S_0^{-1}R = RS_0^{-1}$ is the *largest (two-sided) quotient ring* of R ;
- $\text{Loc}(R, \mathfrak{a}) := \{S^{-1}R = RS^{-1} \mid S \in \text{Den}(R, \mathfrak{a})\}.$

Remark. Subscripts ‘l’ and ‘r’ indicate that left and right versions of a definition/concept are considered respectively.

The largest quotient ring of a ring.

Theorem 4.1 1. For each $\mathfrak{a} \in \text{Ass}(R)$, the set $\text{Den}(R, \mathfrak{a})$ is an ordered abelian semigroup ($S_1 S_2 = S_2 S_1$, and $S_1 \subseteq S_2$ implies $S_1 S_3 \subseteq S_2 S_3$) where the product $S_1 S_2 = \langle S_1, S_2 \rangle$ is the multiplicative subsemigroup of (R, \cdot) generated by S_1 and S_2 .

2. $S_{\mathfrak{a}} := S_{\mathfrak{a}}(R) := \bigcup_{S \in \text{Den}(R, \mathfrak{a})} S$ is the largest element (w.r.t. \subseteq) in $\text{Den}(R, \mathfrak{a})$. The set $S_{\mathfrak{a}}$ is called the largest denominator set associated to \mathfrak{a} .

3. Let $S_i \in \text{Den}(R, \mathfrak{a})$, $i \in I$, where I is an arbitrary non-empty set. Then

$$\langle S_i \mid i \in I \rangle := \bigcup_{\emptyset \neq J \subseteq I, |J| < \infty} \prod_{j \in J} S_j \in \text{Den}(R, \mathfrak{a}) \quad (13)$$

the denominator set generated by the denominators sets S_i , it is the least upper bound of the set $\{S_i\}_{i \in I}$ in $\text{Den}(R, \mathfrak{a})$, i.e. $\langle S_i \mid i \in I \rangle = \bigvee_{i \in I} S_i$.

In Section 5, we will see that for the algebra \mathbb{I}_1 of polynomial integro-differential operators the set $S_0(\mathbb{I}_1)$ (resp. the ring $Q(\mathbb{I}_1)$) is tiny comparing to the sets $S_{l,0}(\mathbb{I}_1)$ and $S_{r,0}(\mathbb{I}_1)$ (resp. to the rings $Q_l(\mathbb{I}_1)$ and $Q_r(\mathbb{I}_1)$).

Corollary 4.2 *The abelian monoid $\text{Den}(R, 0)$ is a complete lattice such that $S_1 S_2 = S_1 \vee S_2$ and $\bigwedge_{i \in I} S_i = \bigvee_{S \in \text{Den}(R, 0), S \subseteq \bigcap_{i \in I} S_i} S$ where all $S_i \in \text{Den}(R, 0)$.*

Clearly, $\bigwedge_{i \in I} S_i$ is the largest element of the set $\{S \mid S \in \text{Den}(R, 0), S \subseteq \bigcap_{i \in I} S_i\}$.

Corollary 4.3 1. *Let R be a ring. Each ring automorphism $\sigma \in \text{Aut}(R)$ of the ring R has the unique extension $\sigma \in \text{Aut}(Q(R))$ to an automorphism of the ring $Q(R)$ given by the rule $\sigma(s^{-1}r) = \sigma(s)^{-1}\sigma(r)$ where $s \in S_0(R)$ and $r \in R$.*

2. *The group $\text{Aut}(R)$ is a subgroup of the group $\text{Aut}(Q(R))$. Moreover, $\text{Aut}(R) = \{\tau \in \text{Aut}(Q(R)) \mid \tau(R) = R\}$.*

Theorem 4.4 1. $S_0(Q(R)) = Q(R)^*$ and $S_0(Q(R)) \cap R = S_0(R)$.

2. $Q(R)^* = \langle S_0(R), S_0(R)^{-1} \rangle$, i.e. the group of units of the ring $Q(R)$ is generated by the sets $S_0(R)$ and $S_0(R)^{-1} := \{s^{-1} \mid s \in S_0(R)\}$.

3. $Q(R)^* = \{s^{-1}t \mid s, t \in S_0(R)\} = \{ts^{-1} \mid s, t \in S_0(R)\}$.

4. $Q(Q(R)) = Q(R)$.

Proof. 1–3. It is obvious that $G := \langle S_0(R), S_0(R)^{-1} \rangle \subseteq Q(R)^* \subseteq S_0(Q(R))$. Applying Proposition 2.7.(2) and its right analogue in the situation where $S = S_0(R)$ and $T = S_0(Q(R))$ we see that

$$S_0(R) \subseteq T' := S_0(Q(R)) \cap R \in \text{Den}_l(R, 0) \cap \text{Den}_r(R, 0),$$

and so $T' \in \text{Den}(R, 0)$ and $S_0(R) = T'$, by the maximality of $S_0(R)$. Let $q \in S_0(Q(R))$. Then $q = s^{-1}t = t_1 s_1^{-1}$ for some elements $s, s_1 \in S_0(R)$, $t, t_1 \in R$, $t = sq \in S_0(Q(R)) \cap R = S_0(R)$ and $t_1 = q s_1 \in S_0(Q(R)) \cap R = S_0(R)$. Therefore, $S_0(Q(R)) \subseteq \{s^{-1}t \mid s, t \in S_0(R)\} \subseteq G$ and $S_0(Q(R)) \subseteq \{ts^{-1} \mid s, t \in S_0(R)\} \subseteq G$, and so $G = Q(R)^* = S_0(Q(R)) = \{s^{-1}t \mid s, t \in S_0(R)\} = \{ts^{-1} \mid s, t \in S_0(R)\}$. This proves statements 1–3.

4. Statement 4 follows from statement 1. \square

Theorem 4.5 *Let R be a ring and \mathcal{C}_R be the set of regular elements of the ring R (i.e. the set of non-zero-divisors). Then the following statements are equivalent.*

1. $Q(R)$ is a semi-simple ring.

2. The rings $Q_{l,cl}(R)$ and $Q_{r,cl}(R)$ exist and are semi-simple.

3. The rings $Q_l(R)$ and $Q_r(R)$ are semi-simple.

If one of the equivalent conditions holds then $S_0(R) = \mathcal{C}_R$ and $Q(R) \simeq Q_{l,cl}(R) \simeq Q_{r,cl}(R) \simeq Q_l(R) \simeq Q_r(R)$.

Proof. (1 \Rightarrow 3) Since $Q(R) \subseteq Q_l(R)$, $Q(R) \subseteq Q_r(R)$ and the ring $Q(R)$ is a semi-simple, by Lemma 3.2.(2), the inclusions are, in fact, equalities (since the regular elements of any semi-simple ring are the units). Therefore, the rings $Q_l(R)$ and $Q_r(R)$ are semi-simple.

(2 \Leftrightarrow 3) Statements 2 and 3 are equivalent, by Theorem 2.9. Moreover, $Q_{l,cl}(R) \simeq Q_{r,cl}(R) \simeq Q_l(R) \simeq Q_r(R)$.

(2 \Rightarrow 1) Recall that $Q_{l,cl}(R) = \mathcal{C}_R^{-1}R$ and $Q_{r,cl}(R) = R\mathcal{C}_R^{-1}$ where \mathcal{C}_R is the set of regular elements of the ring R . Since the rings $Q_{l,cl}(R)$ and $Q_{r,cl}(R)$ exist, they are R -isomorphic. Therefore, $S_0(R) = \mathcal{C}_R$ and $Q(R) \simeq Q_{l,cl}(R) \simeq Q_{r,cl}(R)$. In particular, $Q(R)$ is a semi-simple ring. \square

Proposition 4.6 *If the ring $Q_l(R)$ is a left artinian ring and the ring $Q_r(R)$ is a right artinian ring then $S_0(R) = \mathcal{C}_R$ and $Q(R) \simeq Q_{l,cl}(R) \simeq Q_{r,cl}(R) \simeq Q_l(R) \simeq Q_r(R)$.*

Proof. By Corollary 2.10, $S_{l,0}(R) = \mathcal{C}_R$ and $Q_{l,cl}(R) \simeq Q_l(R)$. By the right version of Corollary 2.10, $S_{r,0}(R) = \mathcal{C}_R$ and $Q_{r,cl}(R) \simeq Q_r(R)$. Then $S_0(R) = \mathcal{C}_R$ and $Q(R) \simeq Q_{l,cl}(R) \simeq Q_{r,cl}(R) \simeq Q_l(R) \simeq Q_r(R)$. \square

The maximal quotients rings of a ring.

Lemma 4.7 *Let $S \in \text{Den}(R, \mathfrak{a})$, $\pi : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, and $\sigma : R \rightarrow S^{-1}R$, $r \mapsto r/1$.*

1. *Let $T \in \text{Den}(S^{-1}R, 0)$ be such that $\pi(S), \pi(S^{-1}) \subseteq T$. Then $T' := \sigma^{-1}(T) \in \text{Den}(R, \mathfrak{a})$, T' is S -saturated, $T = \{s^{-1}t' \mid s \in S, t' \in T'\}$, and $S^{-1}R \subseteq T'^{-1}R = T^{-1}R$.*
2. *$\pi^{-1}(S_0(R/\mathfrak{a})) = S_{\mathfrak{a}}(R)$, $\pi(S_{\mathfrak{a}}(R)) = S_0(R/\mathfrak{a})$ and $Q_{\mathfrak{a}}(R) = Q(R/\mathfrak{a})$.*

Proposition 4.8 *Let $S \in \text{Den}(R, \mathfrak{a})$; $\pi : R \rightarrow R/\mathfrak{a}$, $a \mapsto \bar{a} = a + \mathfrak{a}$; $\sigma : R \rightarrow S^{-1}R$, $r \mapsto r/1$; $G := \langle \pi(S), \pi(S)^{-1} \rangle \subseteq (S^{-1}R)^*$ (i.e. G is the subgroup of the group $(S^{-1}R)^*$ of units of the ring $S^{-1}R$ generated by $\pi(S)^{\pm 1}$). Let $[\sigma^{-1}(G), S_{\mathfrak{a}}(R)] := \{S \in \text{Den}(R, \mathfrak{a}) \mid \sigma^{-1}(G) \subseteq S \subseteq S_{\mathfrak{a}}(R)\}$.*

1. *Let $[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}} := \{S_1 \in [\sigma^{-1}(G), S_{\mathfrak{a}}(R)] \mid \sigma^{-1}(G\pi(S_1)) = S_1\}$ and $[G, S_0(S^{-1}R)] := \{T \in \text{Den}(S^{-1}R, 0) \mid G \subseteq T \subseteq S_0(S^{-1}R)\}$. Then the map*

$$[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}} \rightarrow [G, S_0(S^{-1}R)], \quad S_1 \mapsto \tilde{S}_1 := G\pi(S_1),$$

is an isomorphism of posets and abelian monoids with the inverse map $T \mapsto \sigma^{-1}(T)$ where $G\pi(S_1)$ is the multiplicative monoid generated by G and $\pi(S_1)$ in $S^{-1}R$. In particular,

$$G\pi(S_{\mathfrak{a}}(R)) = S_0(S^{-1}R), \quad S_{\mathfrak{a}}(R) = \sigma^{-1}(S_0(S^{-1}R)), \quad S_{\mathfrak{a}}(R)^{-1}R \simeq S_{\mathfrak{a}}(R)^{-1}R \simeq Q(R/\mathfrak{a}),$$

the monoid $[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}}$ is a complete lattice (since $[G, S_0(S^{-1}R)]$ is a complete lattice, as an interval of the complete lattice $\text{Den}(S^{-1}R, 0)$, Corollary 4.2), and the map $S_1 \mapsto \tilde{S}_1$ is a lattice isomorphism.

2. *Consider the interval $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]$ in $\text{Den}(R/\mathfrak{a}, 0)$. Let $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G\text{-com}} := \{T \in [G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})] \mid GT \cap (R/\mathfrak{a}) = T\}$. Then $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G\text{-com}} \subseteq \text{Den}(S^{-1}R, 0)$ and the map*

$$[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G\text{-com}} \rightarrow [G, S_0(S^{-1}R)], \quad T \mapsto GT,$$

is an isomorphism of posets and abelian monoids with the inverse map $T' \mapsto T' \cap (R/\mathfrak{a})$ where GT is the product in $\text{Den}(S^{-1}R, 0)$. In particular, the monoid $[G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G\text{-com}}$ is a complete lattice.

3. *The map*

$$[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S\text{-com}} \rightarrow [G \cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G\text{-com}}, \quad S_1 \mapsto G\pi(S_1) \cap (R/\mathfrak{a}),$$

is an isomorphism of posets and abelian monoids with the inverse map $S' \mapsto \pi^{-1}(S')$.

Lemma 4.9 1. *For each element $S \in \text{Den}(R)$, there exists a maximal element in the poset $([S, \cdot], \subseteq)$.*

2. *The set $(\max.\text{Den}(R), \subseteq)$ of maximal elements of the poset $(\text{Den}(R), \subseteq)$ is a non-empty set.*

Definition. An element S of the set $\max.\text{Den}(R)$ is called a *maximal denominator set* of R and the ring $S^{-1}R = RS^{-1}$ is called a *maximal quotient ring* of R .

Proposition 4.10 *Let $\mathfrak{a} \in \text{Ass}(R)$, $Q := Q_{\mathfrak{a}}(R)$, Q^* be the group of units of the ring Q and $\sigma : R \rightarrow Q_{\mathfrak{a}}(R)$, $r \mapsto \frac{r}{1}$. Let $T \in \text{Den}(Q, \mathfrak{b})$ where $\mathfrak{b} \in \text{Ass}(Q)$ and Q^*T be the multiplicative sub-semigroup of (Q, \cdot) generated by Q^* and T . Then*

1. *$Q^*T \in \text{Den}(Q, \mathfrak{b})$.*

2. If, in addition, $Q^* \subseteq T$ (eg, Q^*T from statement 1) then

- (a) $T' := \sigma^{-1}(T) \in \text{Den}(R, \mathfrak{b}')$ where $\mathfrak{b}' := \sigma^{-1}(\mathfrak{b}) \supseteq \mathfrak{a}$, $S_{\mathfrak{a}}(R) \subseteq T'$, $T = Q^*\sigma(T')$ (i.e. the monoid T is generated by Q^* and $\sigma(T')$) and $T'^{-1}R = T^{-1}Q$ (i.e. the natural ring monomorphism $T'^{-1}R \rightarrow T^{-1}Q$, $t^{-1}r \mapsto t^{-1}r$, is an isomorphism).
- (b) $S_{\mathfrak{a}}(R) \subseteq S_{\mathfrak{b}'}(R)$ and $S_{\mathfrak{b}'}(R) = \sigma^{-1}(S_{\mathfrak{b}}(Q))$.
- (c) $Q_{\mathfrak{b}'}(R) = Q(Q_{\mathfrak{a}}(R)/\mathfrak{b})$, i.e. the natural ring monomorphism $Q_{\mathfrak{b}'}(R) \rightarrow Q(Q_{\mathfrak{a}}(R)/\mathfrak{b})$, $t^{-1}r \mapsto t^{-1}r$, is an isomorphism.

Let $\max.\text{Ass}(R)$ be the set of maximal elements of the poset $(\text{Ass}(R), \subseteq)$. It is a subset of the set

$$\text{ass.max.Den}(R) := \{\text{ass}(S) \mid S \in \max.\text{Den}(R)\} \quad (14)$$

which is a non-empty set, by Lemma 4.9.(2). Let $\max.\text{Loc}(R)$ be the set of maximal elements of the poset $(\text{Loc}(R), \rightarrow)$. By the very definition of $\text{Loc}(R)$ and by Lemma 4.7.(2),

$$\max.\text{Loc}(R) = \{S^{-1}R \mid S \in \max.\text{Den}(R)\} = \{Q(R/\mathfrak{a}) \mid \mathfrak{a} \in \text{ass.max.Den}(R)\}. \quad (15)$$

Definition. Each element of $\max.\text{Loc}(R)$ is called a *maximal localization ring* (or a *maximal quotient ring*) of the ring R .

Theorem 4.11 *Let $S \in \max.\text{Den}(R)$, $A = S^{-1}R$, A^* be the group of units of the ring A ; $\mathfrak{a} := \text{ass}(S)$, $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, and $\sigma_{\mathfrak{a}} : R \rightarrow A$, $r \mapsto \frac{r}{1}$. Then*

1. $S = S_{\mathfrak{a}}(R)$, $S = \pi_{\mathfrak{a}}^{-1}(S_0(R/\mathfrak{a}))$, $\pi_{\mathfrak{a}}(S) = S_0(R/\mathfrak{a})$ and $A = S_0(R/\mathfrak{a})^{-1}(R/\mathfrak{a}) = (R/\mathfrak{a})S_0(R/\mathfrak{a})^{-1} = Q(R/\mathfrak{a})$.
2. $S_0(A) = A^*$ and $S_0(A) \cap (R/\mathfrak{a}) = S_0(R/\mathfrak{a})$.
3. $S = \sigma_{\mathfrak{a}}^{-1}(A^*)$.
4. $A^* = \langle \pi_{\mathfrak{a}}(S), \pi_{\mathfrak{a}}(S)^{-1} \rangle$, i.e. the group of units of the ring A is generated by the sets $\pi_{\mathfrak{a}}(S)$ and $\pi_{\mathfrak{a}}^{-1}(S) := \{\pi_{\mathfrak{a}}(s)^{-1} \mid s \in S\}$.
5. $A^* = \{\pi_{\mathfrak{a}}(s)^{-1}\pi_{\mathfrak{a}}(t) \mid s, t \in S\}$.
6. $Q(A) = A$ and $\text{Ass}(A) = \{0\}$. In particular, if $T \in \text{Den}(A, 0)$ then $T \subseteq A^*$.

The next theorem is a criterion of when a ring $A \in \text{Loc}(R, \mathfrak{a})$ is equal to $Q_{\mathfrak{a}}(R)$.

Theorem 4.12 *Let $A \in \text{Loc}(R, \mathfrak{a})$, i.e. $A = S^{-1}R$ for some $S \in \text{Den}(R, \mathfrak{a})$. Then $A = Q_{\mathfrak{a}}(R)$ iff $Q(A) = A$.*

Localization maximal rings. We introduce a new class of rings, the localization maximal rings, which turn out to be precisely the class of maximal quotient rings of all rings. As a result, we have a characterization of the maximal quotient rings of a ring (Theorem 4.13).

Definition. A ring A is called a *localization maximal ring* if $A = Q(A)$ and $\text{Ass}(A) = \{0\}$. Clearly, a left and right localization maximal ring A (i.e. $Q_l(A) = A = Q_r(A)$ and $\text{Ass}(A) = \text{Ass}_r(A) = \{0\}$) is a localization maximal but vice versa, in general, as the example of the algebra \mathbb{I}_1 shows (see Section 5 for details).

The next theorem is a criterion of when a quotient ring of a ring is a maximal quotient ring of the ring.

Theorem 4.13 *Let a ring A be a localization of a ring R , i.e. $A \in \text{Loc}(R, \mathfrak{a})$ for some $\mathfrak{a} \in \text{Ass}(R)$. Then $A \in \max.\text{Loc}(R)$ iff $Q(A) = A$ and $\text{Ass}(A) = \{0\}$, i.e. A is a localization maximal ring.*

The next corollary is a criterion of when $S_{\mathfrak{a}_1}(R) \subseteq S_{\mathfrak{a}_2}(R)$ where $\mathfrak{a}_1, \mathfrak{a}_2 \in \text{Ass}(R)$.

Corollary 4.14 *Let $\mathfrak{a}_1, \mathfrak{a}_2 \in \text{Ass}(R)$ and $\sigma_i : R \rightarrow Q_{\mathfrak{a}_i}(R)$, $r \mapsto r/1$, for $i = 1, 2$. Then $S_{\mathfrak{a}_1}(R) \subseteq S_{\mathfrak{a}_2}(R)$ iff $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and $\sigma_2(S_{\mathfrak{a}_1}(R)) \subseteq Q_{\mathfrak{a}_2}(R)^*$.*

Proof. (\Rightarrow) By Lemma 3.2.(2a), $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and, by Lemma 3.2.(2b) and its right analogue, $\sigma_2(S_{\mathfrak{a}_1}(R)) \subseteq Q_{\mathfrak{a}_2}(R)^*$.

(\Leftarrow) Let $S_i := S_{\mathfrak{a}_i}(R)$ and $Q_i := Q_{\mathfrak{a}_i}(R)$ for $i = 1, 2$. Let Q' be the subring of Q_2 generated by R/\mathfrak{a}_2 and $\sigma_2(S_1)^{\pm 1}$. Since $S_1 \in \text{Den}(R, \mathfrak{a}_1)$, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and $\sigma_2(S_1) \subseteq Q_2^*$, every element of Q' has the form $\sigma_2(s)^{-1}\sigma_2(r) = \sigma_2(r')\sigma_2(s')^{-1}$ for some elements $s, s' \in S_1$ and $r, r' \in R$. By the universal property of $Q_1 = S_1^{-1}R = RS_1^{-1}$, there exists a ring homomorphism $Q_1 \rightarrow Q_2$ and so we have the commutative diagram of ring homomorphisms:

$$\begin{array}{ccccc} R & \longrightarrow & R/\mathfrak{a}_1 & \longrightarrow & Q_1 \\ \downarrow = & & \downarrow & & \downarrow \\ R & \longrightarrow & R/\mathfrak{a}_2 & \longrightarrow & Q_2. \end{array}$$

Since $S_i = \sigma_i^{-1}(Q_i^*)$ for $i = 1, 2$ (Lemma 4.7.(2)), we have the inclusion $S_1 \subseteq S_2$. \square

All Ore sets are localizable, a slight generalization of Ore's method of localization.

Definition. An Ore set $S \in \text{Ore}(R)$ is called *localizable* if there exists a ring homomorphism $\varphi : R \rightarrow R'$ where R' is a ring such that, for all $s \in S$, $\varphi(s)$ is a unit of the ring R' .

We will see that all Ore sets are localizable (Theorem 4.15 and Corollary 4.16).

For each right Ore set $S \in \text{Ore}_r(R)$ of the ring R , let $\mathfrak{p}(S)_r := \bigcup_{\alpha \in W} \mathfrak{p}'_{\alpha}$ be the right version of the ideal $\mathfrak{p}(S)_l$ defined in (9) where

$$\mathfrak{p}'_{\alpha+1} := \pi'_{\alpha}{}^{-1}(R/\mathfrak{p}_{\alpha} \cdot \text{ass}_l(\pi'_{\alpha}(S), R/\mathfrak{p}_{\alpha}) + \text{ass}_r(\pi'_{\alpha}(S), R/\mathfrak{p}_{\alpha})), \quad (16)$$

$\pi'_{\alpha} : R \rightarrow R/\mathfrak{p}'_{\alpha}$, $a \mapsto a + \mathfrak{p}'_{\alpha}$. If α is a limit ordinal then

$$\mathfrak{p}'_{\alpha} = \bigcup_{\beta < \alpha} \mathfrak{p}'_{\beta}. \quad (17)$$

For each Ore set $S \in \text{Ore}(R)$, let

$$\mathcal{J}(R, S) := \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ such that } \pi_{\mathfrak{a}}(S) \in \text{Den}(R/\mathfrak{a}, 0)\}$$

where $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$. It follows from (9) that the ideals $\mathfrak{p}(S)_l = \bigcup_{\alpha \in W} \mathfrak{p}_{\alpha}$ and $\mathfrak{p}(S)_r = \bigcup_{\alpha \in W} \mathfrak{p}'_{\alpha}$ coincides since, for all $\alpha \in W$,

$$\mathfrak{p}_{\alpha} = \mathfrak{p}'_{\alpha}. \quad (18)$$

Moreover, for all $\alpha \in W$,

$$\mathfrak{p}_{\alpha+1} := \pi_{\alpha}^{-1}(\text{ass}_l(\pi_{\alpha}(S), R/\mathfrak{p}_{\alpha}) + \text{ass}_r(\pi_{\alpha}(S), R/\mathfrak{p}_{\alpha})) \text{ and } \mathfrak{p}(S) = \bigcup_{\alpha \in W} \mathfrak{p}_{\alpha}. \quad (19)$$

Theorem 4.15 *Let $S \in \text{Ore}(R)$. Then*

1. $\mathcal{J}(R, S) \neq \emptyset$.
2. $\mathfrak{p}(S)$ is the least (with respect to inclusion) element of $\mathcal{J}(R, S)$.
3. If, in addition, $S \in \text{Den}(R, \mathfrak{a})$ then $\mathfrak{p}(S) = \mathfrak{a}$.

Proof. The theorem follows at once from Theorem 3.15 and its analogue for the right Ore sets in R provided we show that $\mathfrak{p}(S) \neq R$. Suppose that $\mathfrak{p}(S) = R$, we seek a contradiction. Then $\mathfrak{p}_\alpha \cap S \neq \emptyset$ for some α . We can assume that α is the least possible. Then α is necessarily not a limit ordinal. Then $0 \notin \pi_{\alpha-1}(S)$ since otherwise we would have $\mathfrak{p}_{\alpha-1} \cap S \neq \emptyset$, a contradiction. Therefore, $\pi_{\alpha-1}(S) \in \text{Ore}(R/\mathfrak{p}_{\alpha-1})$. Replacing R and S by $R/\mathfrak{p}_{\alpha-1}$ and $\pi_{\alpha-1}(S)$ respectively we may assume that $\alpha = 1$, i.e. $s' \in \mathfrak{p}_1 = \text{ass}_l(S, R) + \text{ass}_r(S, R)$ for some $s' \in S$. Then $a + b = s'$ for some elements $a, b \in R$ such that $sa = 0$ and $bt = 0$ for some elements $s, t \in S$. Then $S \ni ss't = s(a + b)t = 0$, a contradiction. The proof of the theorem is complete. \square

Corollary 4.16 *Let S be an Ore set in a ring R . Then there exists an ordered pair (Q, f) where Q is a ring and $f : R \rightarrow Q$ is a ring homomorphism such that*

(i) *for all $s \in S$, $f(s)$ is a unit in Q ;*

and if (Q', f') is another pair satisfying the condition (i) then there is a unique ring homomorphism $h : Q \rightarrow Q'$ such that $f' = hf$. The ring Q is unique up to isomorphism. The ring Q is isomorphic to the localization of the ring $R/\mathfrak{p}(S)$ at the denominator set $\pi(S) \in \text{Den}(R/\mathfrak{p}(S), 0)$ where the ideal $\mathfrak{p}(S)$ of the ring R is defined in (19) and $\pi : R \rightarrow R/\mathfrak{p}(S)$, $a \mapsto a + \mathfrak{p}(S)$.

5 Examples

In this section, the largest (left; right; left and right) quotient ring and the maximal (left; right; left and right) quotient rings are found for the following rings: the endomorphism algebra $\text{End}_K(V)$ of an infinite dimensional vector space with countable basis, semi-prime Goldie rings, the algebra \mathbb{I}_1 of polynomial integro-differential operators, and Noetherian commutative rings.

The endomorphism algebra $\text{End}_K(V)$ of an infinite dimensional vector space V with countable basis. For a vector space V , let

$$\mathcal{F} = \mathcal{F}(V) := \{\varphi \in \text{End}_K(V) \mid \dim_K(\ker(\varphi)) < \infty, \dim_K(\text{coker}(\varphi)) < \infty\}$$

be the set of *Fredholm* linear maps/operators in V .

Lemma 5.1 *Let V be an infinite dimensional vector space with countable basis and $R := \text{End}_K(V)$. Then*

1. *Let $\varphi \in R$ and $\dim_K(\text{im}(\varphi)) = \infty$. Then*
 - (a) $\alpha\varphi\beta = 1$ for some elements α and β of R (necessarily, α is a surjection and β is an injection). Moreover, α and β can be chosen to satisfy the following conditions: $V = \ker(\alpha) \oplus \text{im}(\varphi)$ and $V = \ker(\varphi) \oplus \text{im}(\beta)$.
 - (b) If φ is a surjection then $\varphi\beta = 1$ for some (necessarily, injective) map $\beta \in R$ that can be chosen to satisfy the equality $V = \ker(\varphi) \oplus \text{im}(\beta)$.
 - (c) If φ is an injection then $\alpha\varphi = 1$ for some (necessarily, surjective) map $\alpha \in R$ which can be chosen to satisfy the equality $V = \ker(\alpha) \oplus \text{im}(\varphi)$.
2. *The ideal of compact operators $\mathcal{C} := \{\varphi \in R \mid \dim_K(\text{im}(\varphi)) < \infty\}$ is the only proper ideal of the ring R .*
3. *Let $\varphi \in \mathcal{F}(V)$ and $c \in R$. Then*
 - (a) *If $\varphi c = 0$ then $c \in \mathcal{C}$.*
 - (b) *If $c\varphi = 0$ then $c \in \mathcal{C}$.*
4. *Let $\varphi \in R$. Then*
 - (a) $\ker(\varphi) \neq 0$ iff $\varphi c = 0$ for some element $0 \neq c \in \mathcal{C}$.
 - (b) $\text{im}(\varphi) \neq V$ iff $c\varphi = 0$ for some element $0 \neq c \in \mathcal{C}$.

5. Let $\varphi \in R$. Then φ is a right regular element in R (i.e. $\varphi\psi = 0$ implies $\psi = 0$) iff the map $\varphi : V \rightarrow V$ is an injection.
6. Let $\varphi \in R$. Then φ is a left regular element in R (i.e. $\psi\varphi = 0$ implies $\psi = 0$) iff the map $\varphi : V \rightarrow V$ is a surjection.
7. Let $\varphi \in R$. Then φ is a regular element of R iff φ is a unit of R . So, $\mathcal{C}_R = \text{Aut}_K(V)$ and $Q_{cl}(R) = Q_{cl,l}(R) = Q_{cl,r}(R) = R$.

Proof. 1. This is obvious.

2. Statement 2 follows from statement 1.

3a. $\text{im}(c) \subseteq \ker(\varphi)$, and so $c \in \mathcal{C}$.

3b. $\ker(c) \supseteq \text{im}(\varphi)$, and so $\dim_K(\text{im}(\varphi)) \leq \dim_K(\text{coker}(\varphi)) < \infty$, i.e. $c \in \mathcal{C}$.

4. This is obvious.

5. Statement 5 follows from statement 4a.

6. Statement 6 follows from statement 4b.

7. Statement 7 follows from statements 5 and 6. \square

Theorem 5.2 *Let V be an infinite dimensional vector space with countable basis and $R := \text{End}_K(V)$. Then*

1. $\text{Ass}_l(R) = \text{Ass}_r(R) = \text{Ass}(R) = \{0, \mathcal{C}\}$.
2. $S_{l,0}(R) = S_{r,0}(R) = S_0(R) = \text{Aut}_K(V)$ and $Q_l(R) = Q_r(R) = Q(R) = R$.
3. $S_{l,\mathcal{C}}(R) = S_{r,\mathcal{C}}(R) = S_{\mathcal{C}}(R) = \mathcal{F}$ and $Q_{l,\mathcal{C}}(R) = Q_{r,\mathcal{C}}(R) = Q_{\mathcal{C}}(R) = R/\mathcal{C}$.
4. $\max.\text{Ass}_l(R) = \max.\text{Ass}_r(R) = \max.\text{Ass}(R) = \{\mathcal{C}\}$.
5. R/\mathcal{C} is a localization maximal ring and a left (resp. right; left and right) localization maximal ring.

Proof. 1. Statement 1 follows from statements 2 and 3.

2. Statement 2 follows from Lemma 5.1.(7).

4. Statement 4 follows from statement 3.

3. Let $\pi : R \rightarrow R/\mathcal{C}$, $a \mapsto a + \mathcal{C}$. The ordered pair $(R/\mathcal{C}, \pi)$ satisfies the following conditions

- (i) for all $s \in \mathcal{F}$, $\pi(s)$ is a unit;
- (ii) for all $q \in R/\mathcal{C}$, $q = \pi(s)^{-1}\pi(r) = \pi(r_1)\pi(s_1)^{-1}$ for some $s, s_1 \in \mathcal{F}$ and $r, r_1 \in R$;
- (iii) $\ker(\pi) = \mathcal{C}$ and $\mathcal{C} = \text{ass}_l(\mathcal{F}) = \text{ass}_r(\mathcal{F})$, by Lemma 5.1.(3).

By Ore's Theorem, $R/\mathcal{C} = \mathcal{F}^{-1}R = R\mathcal{F}^{-1}$. To finish the proof of statement 3 it suffices to show that every regular element $\pi(r)$ in R/\mathcal{C} is invertible. Since $\pi(r)$ regular in R/\mathcal{C} , $\dim_K(\ker(r)) < \infty$ (suppose that $\dim_K(\ker(r)) = \infty$, we seek a contradiction. Fix a complement subspace U of $\ker(r)$ in V , i.e. $V = \ker(r) \oplus U$. Let p be the projection onto $\ker(r)$. Then $\pi(r)\pi(p) = 0$ but $\pi(p) \neq 0$ since $\dim_K(\ker(r)) = \infty$, a contradiction).

Similarly, since $\pi(r)$ is regular in R/\mathcal{C} , $\dim_K(\text{coker}(r)) < \infty$ (suppose that $\dim_K(\text{coker}(r)) = \infty$, we seek a contradiction. Fix a complement subspace W of $\text{im}(r)$ in V , i.e. $V = \text{im}(r) \oplus W$. Let q be the projection onto W . Then $\pi(q)\pi(r) = 0$ but $\pi(q) \neq 0$ since $\dim_K(W) = \dim_K(\text{coker}(r)) = \infty$, a contradiction). Therefore, $r \in \mathcal{F}$, and so $\pi(r)$ is a unit.

5. Statement 5 follows from statements 3 and 4, see Theorem 3.11. \square

Semi-prime Goldie rings. A ring R is called a *left Goldie* ring if R has finite left uniform dimension and R satisfies the a.c.c. on left annihilators. Similarly, a right Goldie ring is defined. A left and right Goldie ring is called a *Goldie* ring. The reader is referred to the books [3], [5] and [6] for more details.

Lemma 5.3 *Let R be a ring.*

1. If S and T are left (resp. right; left and right) Ore sets in R then so is the semigroup ST generated by S and T in R provided $0 \notin ST$.
2. If $S \in \text{Den}(R, \mathfrak{a})$ and $C \in \text{Den}(R, 0)$ then $CS \in \text{Den}(R, \mathfrak{a})$, i.e. the monoid CS generated by C and S in R is a denominator set with $\text{ass}(CS) = \mathfrak{a}$.

Proof. 1. It suffices to prove statement 1, say, for left Ore sets S and T . We have to show that for elements $s_1 t_1 s_2 t_2 \cdots s_n t_n \in ST$ (where $s_i \in S$ and $t_i \in T$) and $r \in R$ there exist elements $\theta \in ST$ and $r' \in R$ such that $\theta r = r' s_1 t_1 \cdots s_n t_n$. Since T is a left Ore set, there are elements $t'_n \in T$ and $r'_n \in R$ such that $t'_n r = r'_n t_n$. Since S is a left Ore set, there are elements $s'_n \in S$ and $r''_n \in R$ such that $s'_n r'_n = r''_n s_n$. Therefore, $s'_n t'_n r = r''_n s_n t_n$. Repeating these two steps $n - 1$ more times we find elements $s'_1, \dots, s'_{n-1} \in S$; $t'_1, \dots, t'_{n-1} \in T$ and $r''_1 \in R$ such that $s'_1 t'_1 \cdots s'_{n-1} t'_{n-1} r = r''_1 s_1 t_1 \cdots s_n t_n$. Now, set $\theta := s'_1 t'_1 \cdots s'_{n-1} t'_{n-1} \in S$ and $r' := r''_1 \in R$.

2. Let us show that CS is a multiplicative set. Suppose that $c_1 s_1 \cdots c_n s_n = 0$ for some elements $c_i \in C$ and $s_i \in S$, we seek a contradiction. Then $s_1 c_2 s_2 \cdots c_n s_n = 0$ since c_1 is a regular element, and so $c_2 s_2 \cdots c_n s_n s'_n = 0$ for some element $s'_n \in S$ since $S \in \text{Den}(R, \mathfrak{a})$. Repeating this argument several times we come to the situation where $c' s' = 0$ for some elements $c' \in C$ and $s' \in S$, i.e. $s' = 0$ (since c' is a regular element), a contradiction.

By statement 1, CS is an Ore set in R . To finish the proof of statement 2 it suffices to show that $\text{ass}(CS) = \mathfrak{a}$, i.e. $c_1 s_1 \cdots c_n s_n r = 0$ for some $c_i \in C$, $s_i \in S$ and $r \in R$, implies that $r \in \mathfrak{a}$. The element c_1 is a regular element, hence $s_1 \cdots c_n s_n r = 0$, and so $c_2 s_2 \cdots c_n s_n r s'_1 = 0$ for some $s'_1 \in S$ since $S \in \text{Den}(R, \mathfrak{a})$. Repeating the same two steps $n - 1$ more times gives $r s'_1 s'_2 \cdots s'_n = 0$ for some elements $s'_i \in S$, i.e. $r \in \mathfrak{a}$. \square

Corollary 5.4 *Let R be a prime Goldie ring and \mathcal{C}_R be the set of regular elements of R . Then $\text{Ass}(R) = \{0\}$, $S_0(R) = \mathcal{C}_R$, $Q_0(R) = Q_{cl}(R)$ is the only maximal localization of the ring R .*

Proof. Let $\mathfrak{a} \in \text{Ass}(R)$ and $S \in \text{Den}(R, \mathfrak{a})$. By Lemma 5.3.(2), $\mathcal{C}_R S \subseteq S_{\mathfrak{a}}(R)$. Since $Q_{cl}(R)$ is a simple artinian ring (i.e. the matrix ring over a division ring), we see that $\mathfrak{a} = 0$, and so $\mathcal{C}_R = S_0(R)$. \square

Let R be a semi-prime Goldie ring which is not a prime ring and \mathcal{C}_R be its set of regular elements. Let $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the set of minimal primes of the ring R . By Goldie's Theorem, $Q_{cl}(R) := \mathcal{C}_R^{-1} R \simeq R \mathcal{C}_R^{-1} \simeq \prod_{i=1}^s R_i$ is the direct product of simple artinian rings R_i (i.e. R_i is a matrix ring over a division ring). The ring R can be identified with its image under the ring monomorphism $\sigma : R \rightarrow Q_{cl}(R)$, $r \mapsto r/1$. For each $i = 1, \dots, s$, $\mathfrak{p}_i = \sigma^{-1}(\prod_{j \neq i} R_j)$ (Proposition 3.2.2, [5]). For each non-empty set I of the set $\{1, \dots, s\}$, let the ring homomorphism $\sigma_I : R \rightarrow R_I := \prod_{i \in I} R_i$ be the composition of σ and the projection $\prod_{i=1}^s R_i \rightarrow R_I$. Let R_i^* be the group of units of the ring R_i . Then $R_I^* = \prod_{i \in I} R_i^*$ is the group of units of the ring R_I . A subset A of a set B is called a *proper subset* of B if $A \neq \emptyset, B$.

Theorem 5.5 *Let R be a semi-prime Goldie ring which is not a prime ring and $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the set of its minimal prime ideals. Then*

1. $\text{Ass}(R) := \{\mathfrak{a}(I) := \bigcap_{i \in I} \mathfrak{p}_i \mid \emptyset \subsetneq I \subseteq \{1, \dots, s\}\}$.
2. For each proper subset I of $\{1, \dots, s\}$, $S_{\mathfrak{a}(I)}(R) = \sigma_I^{-1}(R_I^* \times R_{CI})$ and $Q_{\mathfrak{a}(I)}(R) = Q_{cl}(R)/R_{CI} \simeq R_I$ where $CI := \{1, \dots, s\} \setminus I$.
3. $S_0(R) = \mathcal{C}_R \subseteq S_{\mathfrak{a}}(R)$ for all $\mathfrak{a} \in \text{Ass}(R)$.
4. $S_{\mathfrak{a}(I)}(R) \subseteq S_{\mathfrak{a}(J)}(R)$ iff $I \supseteq J$ where I and J are proper subsets of $\{1, \dots, s\}$.
5. If I and J are proper subsets of $\{1, \dots, s\}$ such that $I \supseteq J$ then, by statement 4 and the universal property of localization, there is the unique ring homomorphism $Q_{\mathfrak{a}(I)}(R) = \prod_{i \in I} R_i \rightarrow Q_{\mathfrak{a}(J)}(R) = \prod_{j \in J} R_j$ which is necessarily the projection onto $\prod_{j \in J} R_j$ in $\prod_{i \in I} R_i$.

6. $\max.\text{Ass}(R) = \{\mathfrak{p}_i, | i = 1, \dots, s\}$, $\{Q_{\mathfrak{p}_i}(R) = R_i | i = 1, \dots, s\}$ is the set of maximal localizations of the ring R and $\text{ass}(Q_{\mathfrak{p}_i})(R) = \mathfrak{p}_i$ for all $i = 1, \dots, s$.

Proof. 1–3. By Theorem 4.5, $S_0(R) = \mathcal{C}_R$ and $Q(R) = Q_{cl}(R)$. By Lemma 5.3.(2), $\mathcal{C}_R \subseteq S_{\mathfrak{a}}(R)$ for all $\mathfrak{a} \in \text{Ass}(R)$ (and statement 3 follows), and there is a natural homomorphism $Q_{cl}(R) \rightarrow Q_{\mathfrak{a}}(R)$, $r \mapsto r/1$. By Lemma 3.2, the ring $Q_{\mathfrak{a}}(R)$ is a localization of the ring $Q_{cl}(R)$. Moreover, by Proposition 4.10.(2), the map $\text{Ass}(R) \rightarrow \text{Ass}(Q_{cl}(R))$, $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$, is a bijection with the inverse map

$$\mathfrak{b} \mapsto R \cap \mathfrak{b}, \quad (20)$$

and the map

$$\{S_{\mathfrak{a}}(R) | \mathfrak{a} \in \text{Ass}(R)\} \rightarrow \{S_{\mathfrak{b}}(Q_{cl}(R)) | \mathfrak{b} \in \text{Ass}(Q_{cl}(R))\}, \quad S_{R \cap \mathfrak{b}}(R) \mapsto S_{\mathfrak{b}}(Q_{cl}(R)),$$

is a bijection and

$$S_{R \cap \mathfrak{b}}(R) = \sigma_{R \cap \mathfrak{b}}^{-1}(S_{\mathfrak{b}}(Q_{cl}(R))) \quad (21)$$

where

$$\sigma_{R \cap \mathfrak{b}} : R \rightarrow Q_{R \cap \mathfrak{b}}(R) = S_{R \cap \mathfrak{b}}(R)^{-1}R \simeq S_{\mathfrak{b}}(Q_{cl}(R))^{-1}Q_{cl}(R). \quad (22)$$

Clearly, $\text{Ass}(Q_{cl}(R)) = \text{Ass}(\prod_{i=1}^s R_i) = \{0\} \cup \{R_I, | \emptyset \neq I \subsetneq \{1, \dots, s\}\}$ and $R_I = Q_{cl}(R)/R_{CI} = Q_{R_{CI}}(Q_{cl}(R))$ (by Theorem 4.12, since every regular element of R_I is a unit). Now, statement 1 follows from (20) and the fact that $\mathfrak{p}_i = R \cap \prod_{j \neq i} R_j$. For every proper subset I of $\{1, \dots, s\}$, $R \cap R_I = \mathfrak{a}(CI)$, $\sigma_{R \cap R_I} = \sigma_{CI}$ and $S_{R_I}(Q_{cl}(R)) = R_{CI}^* \times R_I$. Then, by (21), $S_{R \cap R_I}(R) = \sigma_{R \cap R_I}^{-1}(S_{R_I}(Q_{cl}(R))) = \sigma_{CI}^{-1}(R_{CI}^*)$, and, by (22), $Q_{\mathfrak{a}(CI)}(R) = Q_{R \cap R_I}(R) \simeq S_{R_I}(Q_{cl}(R))^{-1}Q_{cl}(R) \simeq R_{CI}$.

4. Statement 4 follows from statement 2.
5. Statement 5 follows from statement 2.
6. Statement 6 follows from statements 2, 4 and 5. \square

The algebra \mathbb{I}_1 of polynomial integro-differential operators. Let us collect some facts for the algebra \mathbb{I}_1 which are used in the proofs of Theorem 5.7 and Proposition 5.8. For the details the reader is referred to [1] or [2]. Throughout, K is a field of characteristic zero and K^* is its group of units; $P_1 := K[x]$ is a polynomial algebra in one variable x over K ; $\partial := \frac{d}{dx}$; $\text{End}_K(P_1)$ is the algebra of all K -linear maps from P_1 to P_1 , and $\text{Aut}_K(P_1)$ is its group of units (i.e. the group of all the invertible linear maps from P_1 to P_1); the subalgebras $A_1 := K\langle x, \partial \rangle$ and $\mathbb{I}_1 := K\langle x, \partial, \int \rangle$ of $\text{End}_K(P_1)$ are called the (first) *Weyl algebra* and the *algebra of polynomial integro-differential operators* respectively where $\int : P_1 \rightarrow P_1$, $p \mapsto \int p dx$, is the *integration*, i.e. $\int : x^n \mapsto \frac{x^{n+1}}{n+1}$ for all $n \in \mathbb{N}$. The algebra \mathbb{I}_1 is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals, [1].

The algebra \mathbb{I}_1 is generated by the elements ∂ , $H := \partial x$ and \int (since $x = \int H$) that satisfy the defining relations (Proposition 2.2, [1]):

$$\partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra \mathbb{I}_1 ,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \quad (23)$$

satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00} \partial^j$.

The algebra $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$ is a \mathbb{Z} -graded algebra ($\mathbb{I}_{1,i}\mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|} D_1 = D_1 \partial^{|i|} & \text{if } i < 0, \end{cases} \quad (24)$$

the algebra $D_1 := K[H] \oplus \bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of \mathbb{I}_1 , $He_{ii} = e_{ii}H = (i+1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals of D_1); $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; $_{D_1}(D_1 \partial^i) \simeq D_1$, $d \partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \rightarrow D_1$, $d \mapsto d \int^i$, and $\partial^i \cdot : D_1 \rightarrow D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$.

Each element a of the algebra \mathbb{I}_1 is the unique finite sum

$$a = \sum_{i>0} a_{-i} \partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (25)$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [1]. The algebra \mathbb{I}_1 has the only proper ideal $F = \bigoplus_{i,j \in \mathbb{N}} Ke_{ij} \simeq M_\infty(K)$ and $F^2 = F$. The factor algebra \mathbb{I}_1/F is canonically isomorphic to the skew Laurent polynomial algebra $B_1 := K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H+1$, via $\partial \mapsto \partial$, $\int \mapsto \partial^{-1}$, $H \mapsto H$ (where $\partial^{\pm 1} \alpha = \tau^{\pm 1}(\alpha) \partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1}H$). Therefore, they have the common skew field of fractions, $\text{Frac}(A_1) = \text{Frac}(B_1)$, the *first Weyl skew field*.

The algebra \mathbb{I}_1 admits the involution $*$ over the field K : $\partial^* = \int$, $\int^* = \partial$ and $H^* = H$, i.e. it is a K -algebra *anti-isomorphism* $((ab)^* = b^* a^*)$ such that $a^{**} = a$. Therefore, the algebra \mathbb{I}_1 is *self-dual*, i.e. it is isomorphic to its opposite algebra \mathbb{I}_1^{op} . As a result, the left and right properties of the algebra \mathbb{I}_1 are the same. Clearly, $e_{ij}^* = e_{ji}$ for all $i, j \in \mathbb{N}$, and so $F^* = F$.

The next theorem describes the one-sided largest quotient rings of the algebra \mathbb{I}_1 .

Theorem 5.6 (Theorem 9.7, [2])

1. $S_{r,0}(\mathbb{I}_1) = \mathbb{I}_1 \cap \text{Aut}_K(K[x])$ and the largest regular right quotient ring $Q_r(\mathbb{I}_1)$ of \mathbb{I}_1 is the subalgebra of $\text{End}_K(K[x])$ generated by \mathbb{I}_1 and $S_{r,0}(\mathbb{I}_1)^{-1} := \{s^{-1} \mid s \in S_{r,0}(\mathbb{I}_1)\}$.
2. $S_{l,0}(\mathbb{I}_1) = S_{r,0}(\mathbb{I}_1)^*$ and $S_{l,0}(\mathbb{I}_1) \neq S_{r,0}(\mathbb{I}_1)$.
3. The rings $Q_l(\mathbb{I}_1)$ and $Q_r(\mathbb{I}_1)$ are not isomorphic.

The next theorem describes the (two-sided) largest quotient ring of the algebra \mathbb{I}_1 , it is tiny comparing with the one-sided largest quotient rings of the algebra \mathbb{I}_1 .

Theorem 5.7 Let $\mathcal{M} := (K[H] + F) \cap \text{Aut}_K(K[x])$. Then

1. $S_0(\mathbb{I}_1) = S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1)$, $S_0(\mathbb{I}_1)$ is a proper subset of the sets $S_{l,0}(\mathbb{I}_1)$ and $S_{r,0}(\mathbb{I}_1)$, and $S_0(\mathbb{I}_1)^* = S_0(\mathbb{I}_1)$ where $*$ is the involution of the algebra \mathbb{I}_1 .
2. $S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1) = \mathcal{M}$ and \mathcal{M} is the set of regular elements of the algebra $K[H] + F$.
3. Let $\mathcal{M}_0 := D_1 \cap \text{Aut}_K(K[x])$. Then $\mathcal{M}_0 \subseteq \mathcal{M}$, $\mathcal{M} = \mathcal{M}_0(1+F)^* = (1+F)^* \mathcal{M}_0$ and $\mathcal{M}_0 \cap (1+F)^* = (1+F_0)^*$ where $F_0 := \bigoplus_{i \in \mathbb{N}} Ke_{ii}$.
4. \mathcal{M}_0 is the set of regular elements of the commutative (non-Noetherian) algebra D_1 ; $D_1 = \mathcal{M}_0(1+F_0) \coprod F_0 = \mathcal{M}_0 \cup \{0\} + F_0$, $Q_{cl}(D_1) := \mathcal{M}_0^{-1} D_1 = \mathcal{M}_0^{-1} \mathcal{M}_0(1+F_0) \coprod F_0 = \mathcal{M}_0^{-1} \mathcal{M}_0 \cup \{0\} + F_0$.
5. $Q(\mathbb{I}_1) = S_0(\mathbb{I}_1)^{-1} \mathbb{I}_1 = \sum_{i \in \mathbb{Z}} Q_{cl}(D_1) v_i + F = \sum_{i \in \mathbb{Z}} (\mathcal{M}_0^{-1} \mathcal{M}_0 \cup \{0\}) v_i + F = \sum_{i \in \mathbb{Z}} v_i Q_{cl}(D_1) + F = \sum_{i \in \mathbb{Z}} v_i (\mathcal{M}_0^{-1} \mathcal{M}_0 \cup \{0\}) + F$ where $Q_{cl}(D_1)$ is the classical ring of fractions of the commutative ring D_1 and

$$v_i := \begin{cases} \int^i & \text{if } i \geq 1, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i \leq -1. \end{cases}$$

6. $Q(\mathbb{I}_1) \subsetneq Q_l(\mathbb{I}_1)$ and $Q(\mathbb{I}_1) \subsetneq Q_r(\mathbb{I}_1)$.

Proof. 2. Recall that $S_{r,0}(\mathbb{I}_1) = \mathbb{I}_1 \cap \text{Aut}_K(K[x])$ (Theorem 9.7.(2), [2]), $S_{l,0}(\mathbb{I}_1) = S_{r,0}(\mathbb{I}_1)^*$ (Theorem 9.7.(3), [2]), and if $a \in \mathbb{I}_1 \setminus F$ and $a \in \mathbb{I}_1 \cap \text{Aut}_K(K[x])$ then $a \in \sum_{i \leq 0} D_1 v_i + F$ (Theorem 6.2.(1), [2]). Since $(v_i)^* = v_{-i}$ and $d^* = d$ for all elements $d \in D_1$,

$$\begin{aligned} S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1) &= S_{l,0}(\mathbb{I}_1) \cap S_{l,0}(\mathbb{I}_1)^* \subseteq (K[H] + F) \cap \text{Aut}_K(K[x]) \\ &\subseteq S_{l,0}(\mathbb{I}_1) \cap S_{l,0}(\mathbb{I}_1)^* = S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1) \end{aligned}$$

i.e. $S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1) = \mathcal{M}$.

Let \mathcal{R} be the set of regular elements of the algebra $K[H] + F$. Clearly, $\mathcal{M} \subseteq \mathcal{R}$. Let $r \in \mathcal{R}$. Then $r \notin F$. By Proposition 6.1.(1), [2], $\text{ind}_{K[x]}(r) = 0$. Then, by Theorem 6.2.(3), [2], $r_{K[x]}$ is a bijection iff $r_{K[x]}$ is an injection iff $r_{K[x]}$ is a surjection where $r_{K[x]} : K[x] \rightarrow K[x]$, $p \mapsto rp$. Therefore, to prove the equality $\mathcal{M} = \mathcal{R}$ it suffices to show that $r_{K[x]}$ is an injection. Suppose that $\ker(r_{K[x]}) \neq 0$, we seek a contradiction. Since $r \notin F$, by Lemma 6.10, [2], there exists an idempotent $f \in F$ such that $\ker(r_{K[x]}) = \text{im}(f_{K[x]})$. In particular, $rf = 0$ but $f \neq 0$, a contradiction. Therefore, $\mathcal{M} = \mathcal{R}$.

1. Since $S_{l,0}(\mathbb{I}_1) \neq S_{r,0}(\mathbb{I}_1)$ (Theorem 9.7.(3,4), [2]), the set $S_0(\mathbb{I}_1)$ is a proper subset of the sets $S_{l,0}(\mathbb{I}_1)$ and $S_{r,0}(\mathbb{I}_1)$. Clearly,

$$\begin{aligned} S_0(\mathbb{I}_1) &\subseteq \mathcal{M} = S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1) = S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1)^* \\ &= (S_{l,0}(\mathbb{I}_1) \cap S_{l,0}(\mathbb{I}_1)^*)^* = S_{l,0}(\mathbb{I}_1) \cap S_{r,0}(\mathbb{I}_1) = \mathcal{M}. \end{aligned}$$

We have used the fact that $S_{l,0}(\mathbb{I}_1) = S_{r,0}(\mathbb{I}_1)^*$ (Theorem 9.7.(3), [2]). In view of statement 2 and the equality $\mathcal{M}^* = \mathcal{M}$, to finish the proof of statement 1 it suffices to show that \mathcal{M} is a left Ore set in \mathbb{I}_1 , that is, for any elements $a \in \mathbb{I}_1$ and $u = \alpha + u_1 \in \mathcal{M}$ where $\alpha \in K[H]$ and $u_1 \in F$, there exists an element $b \in \mathcal{M}$ such that $bau^{-1} \in \mathbb{I}_1$ (where the product bau^{-1} is taken in $\text{End}_K(K[x])$, recall that $\mathbb{I}_1 \subseteq \text{End}_K(K[x])$). The element a is a sum $\sum_{i=-n}^n \alpha_i v_i + f$ for some natural number n where $\alpha_i \in K[H]$.

For all elements $\beta \in K[H]$, $\beta v_i = v_i \tau^i(\beta)$ where $\tau \in \text{Aut}_K(K[H])$ and $\tau(H) = H + 1$. Notice that $F\mathcal{M}^{-1} \subseteq F$ and $\alpha \cdot u^{-1} = \alpha(\alpha + u_1)^{-1} \in (1 + F)^* \subseteq \mathcal{M}$. Let $b' := \prod_{i=-n}^n \tau^i(\alpha)$. Then

$$b'au^{-1} = b' \left(\sum_{i=-n}^n \alpha_i v_i u^{-1} + f u^{-1} \right) = \sum_{i=-n}^n \alpha_i v_i \frac{\tau^i(b')}{\alpha} \alpha u^{-1} + b' f u^{-1} \in \mathbb{I}_1,$$

since $\frac{\tau^i(b')}{\alpha} \in K[H]$, $\alpha u^{-1} \in \mathcal{M}$ and $b' f u^{-1} \in b' \cdot F\mathcal{M}^{-1} \subseteq b' \cdot F \subseteq F$. Let $I := \{i \in \mathbb{N} \mid b * x^i = 0\}$. Since $H * x^i = (i+1)x^i$ for all $i \in \mathbb{N}$, we see that $I = \{i \in \mathbb{N} \mid i+1 \text{ is a root of the polynomial } b' \in K[H]\}$. Then the element $b := b' + \sum_{i \in I} e_{ii} \in \mathcal{M}$ and $bau^{-1} \in \mathbb{I}_1$.

3. Since $D_1 = K[H] + F_0 \subseteq K[H] + F$, the inclusion $\mathcal{M}_0 \subseteq \mathcal{M}$ follows. It is obvious that $\mathcal{M}_0 \cap (1 + F)^* = (1 + F_0)^*$. To prove the equality $\mathcal{M} = \mathcal{M}_0(1 + F)^*$ we have to show that each element $u \in \mathcal{M}$ is a product vw for some elements $v \in \mathcal{M}_0$ and $w \in (1 + F)^*$. Notice that $u = \alpha + f$ for some elements $\alpha \in K[H]$ and $f \in F$. Choose an element, say $g \in F_0$, such that $v := \alpha + g \in \mathcal{M}_0$ (see the proof of statement 1). Then $u = \alpha + g + f - g = v(1 + v^{-1}(f - g)) = vw$ where $w := 1 + v^{-1}(f - g) \in (1 + F)^*$ since $\mathcal{M}_0^{-1}F \subseteq F$. Therefore, $\mathcal{M} = \mathcal{M}_0(1 + F)^*$. Then $\mathcal{M} = (1 + F)^* \mathcal{M}_0$ since, for all elements $s \in \mathcal{M}_0$, $s(1 + F)^* s^{-1} \subseteq (1 + F)^*$.

4. Let \mathcal{R} be the set of regular elements of the ring D_1 . Then $\mathcal{M}_0 \subseteq \mathcal{R}$. Let $r \in \mathcal{R}$, in order to prove that the inverse inclusion holds, $\mathcal{M}_0 \supseteq \mathcal{R}$, we have to show that $r \in \mathcal{M}_0$ or, equivalently, that $r \in \text{Aut}_K(K[x])$. Notice that $K[x] = \bigoplus_{i \in \mathbb{N}} Kx^i$ is the direct sum of r -invariant subspaces, i.e. $r * Kx^i \subseteq Kx^i$ for all $i \in \mathbb{N}$. If $r \notin \text{Aut}_K(K[x])$ then $r * Kx^i = 0$ for some i , and so $re_{ii} = 0$ since $\text{im}(e_{ii}) = Kx^i$ where $e_{ii} \in D_1$, a contradiction. Therefore, $r \in \text{Aut}_k(K[x])$, and so $\mathcal{M}_0 = \mathcal{R}$.

Clearly, $D_1 \supseteq \mathcal{M}_0(1 + F_0) \amalg F_0$. To prove that the reverse inclusion holds we have to show that every element $d \in D_1 \setminus F_0$ belongs to the set $\mathcal{M}_0(1 + F_0)$. Notice that $d = \alpha + f$ for some elements $0 \neq \alpha \in K[H]$ and $f \in F_0$. Choose an element, say $g \in F_0$, such that $u := \alpha + g \in \mathcal{M}_0$.

Then $d = \alpha + g + f - g = u(1 + u^{-1}(f - g)) \in \mathcal{M}_0(1 + F_0)$ since $\mathcal{M}_0^{-1}F_0 \subseteq F_0$. Therefore, $D_1 = \mathcal{M}_0(1 + F_0) \coprod F_0$, and so $D_1 = \mathcal{M}_0 \cup \{0\} + F_0$. Then $\mathcal{M}_0^{-1}D_1 = \mathcal{M}_0^{-1}\mathcal{M}(1 + F) \coprod F_0 = \mathcal{M}_0^{-1}\mathcal{M} \cup \{0\} + F_0$ since $\mathcal{M}^{-1}F_0 \subseteq F_0$.

5. By statements 3 and 4, for all $i \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{M}^{-1}D_1v_i &= \mathcal{M}_0^{-1}(1 + F)^*D_1v_i = \mathcal{M}_0^{-1}(1 + F)^*(\mathcal{M}_0 \cup \{0\} + F_0)v_i \\ &\subseteq (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\} + F)v_i \subseteq (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\})v_i + F. \end{aligned}$$

Therefore,

$$\begin{aligned} S_0(\mathbb{I}_1)^{-1}\mathbb{I}_1 &= \mathcal{M}^{-1}\mathbb{I}_1 = \sum_{i \in \mathbb{Z}} \mathcal{M}^{-1}D_1v_i + \mathcal{M}^{-1}F = \sum_{i \in \mathbb{Z}} (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\})v_i + F \\ &= \sum_{i \in \mathbb{Z}} (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\} + F_0)v_i + F = \sum_{i \in \mathbb{Z}} Q_{cl}(D_1)v_i + F. \end{aligned}$$

Applying the involution $*$ to the inclusion $\mathcal{M}^{-1}D_1v_i \subseteq (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\})v_i + F$ and using the equalities $\mathcal{M}^* = \mathcal{M}$, $D_1^* = D_1$ and $v_i^* = v_{-i}$ we obtain the inclusion $v_{-i}D_1\mathcal{M}^{-1} \subseteq v_{-i}(\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\}) + F$ for all $i \in \mathbb{Z}$. Then

$$\begin{aligned} \mathbb{I}_1 S_0(\mathbb{I}_1)^{-1} &= \mathbb{I}_1 \mathcal{M}^{-1} = \sum_{i \in \mathbb{Z}} v_i D_1 \mathcal{M}^{-1} + F \mathcal{M}^{-1} = \sum_{i \in \mathbb{Z}} v_i (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\}) + F \\ &= \sum_{i \in \mathbb{Z}} v_i (\mathcal{M}_0^{-1}\mathcal{M}_0 \cup \{0\} + F) + F = \sum_{i \in \mathbb{Z}} v_i Q_{cl}(D_1) + F. \end{aligned}$$

6. The inclusion $Q(\mathbb{I}_1) \subseteq Q_l(\mathbb{I}_1)$ (resp. $Q(\mathbb{I}_1) \subseteq Q_r(\mathbb{I}_1)$) is a proper inclusion by statement 5 and Theorem 9.7, [2]. Another way to prove this is as follows. Suppose that $Q(\mathbb{I}_1) = Q_l(\mathbb{I}_1)$, we seek a contradiction. The algebra \mathbb{I}_1 has the involution $*$, and so it is isomorphic to its opposite algebra of \mathbb{I}_1 . Therefore, $Q(\mathbb{I}_1) = Q_r(\mathbb{I}_1)$, but the rings $Q_l(\mathbb{I}_1)$ and $Q_r(\mathbb{I}_1)$ are not isomorphic, by Theorem 9.7.(4), [2], a contradiction. \square

Proposition 5.8 1. $\text{Ass}_l(\mathbb{I}_1) = \text{Ass}_r(\mathbb{I}_1) = \text{Ass}(\mathbb{I}_1) = \{0, F\}$ and $\max.\text{Ass}_l(\mathbb{I}_1) = \max.\text{Ass}_r(\mathbb{I}_1) = \max.\text{Ass}(\mathbb{I}_1) = \{F\}$.

2. $S_{l,F}(\mathbb{I}_1) = S_{r,F}(\mathbb{I}_1) = S_F(\mathbb{I}_1) = \mathbb{I}_1 \setminus F$ and $Q_{l,F}(\mathbb{I}_1) = Q_{r,F}(\mathbb{I}_1) = Q_F(\mathbb{I}_1) = \text{Frac}(B_1) = \text{Frac}(A_1)$.

3. $\max.\text{Den}_l(\mathbb{I}_1) = \max.\text{Den}_r(\mathbb{I}_1) = \max.\text{Den}(\mathbb{I}_1) = \{\mathbb{I}_1 \setminus F\}$.

Proof. 2. Let $\pi_F : \mathbb{I}_1 \rightarrow \mathbb{I}_1/F = B_1$, $a \mapsto a + F$. Then $\pi_F^{-1}(B_1 \setminus \{0\}) = \mathbb{I}_1 \setminus F$ is a multiplicative set of the algebra \mathbb{I}_1 with $\text{ass}_l(\mathbb{I}_1 \setminus F) = \text{ass}_r(\mathbb{I}_1 \setminus F) = F$ since for all $i, j \in \mathbb{N}$, $\partial^{i+1}e_{ij} = 0$, $e_{ij} \int^{i+1} = 0$ and $F = \bigoplus_{i,j \in \mathbb{N}} K e_{ij}$ is the only proper ideal of the algebra \mathbb{I}_1 and the algebra $\mathbb{I}_1/F = B_1$ is a domain. Since B_1 is a Noetherian domain and $\text{Frac}(A_1) = \text{Frac}(B_1) = (B_1 \setminus \{0\})^{-1}B_1 = B_1(B_1 \setminus \{0\})^{-1}$, we see that, by the universal property of localization, $\text{Frac}(\mathbb{I}_1) = (\mathbb{I}_1 \setminus F)^{-1}\mathbb{I}_1 = \mathbb{I}_1(\mathbb{I}_1 \setminus F)^{-1}$ and so $\mathbb{I}_1 \setminus F$ is a left and right denominators set of the algebra \mathbb{I}_1 with $\text{ass}(\mathbb{I}_1 \setminus F) = F$. Now, statement 2 follows from Theorem 3.9.(1).

1. Statement 1 follows from statement 2.

3. Statement 3 follows from statements 1 and 2. \square

Noetherian commutative rings.

Example. If R is a commutative ring with a single minimal prime ideal \mathfrak{p} (eg, R is a commutative domain) then $\text{Ass}(R) = \{0\}$, $S_0(R) = R \setminus \mathfrak{p}$ and $Q(R) = R_{\mathfrak{p}}$, the localization of the ring R at the prime ideal \mathfrak{p} .

Lemma 5.9 Let R be a commutative ring, C_R be its set of regular elements and $S \in \text{Ore}(R)$. Then

1. $\mathfrak{p}(S) = \text{ass}(S)$.
2. $\mathcal{C}_R S \in \text{Den}(R, \text{ass}(S))$.

Proof. 1. The equality follows from the definition of the ideal $\mathfrak{p}(S)$, see (18) and (19).
 2. Obvious. \square

Let R be a Noetherian commutative ring and $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be its set of minimal primes with $s \geq 2$. Then $S_0 = S_0(R) = R \setminus \bigcup_{i=1}^s \mathfrak{p}_i$ is the set of regular elements of the ring R (i.e. the set of non-zero-divisors of R) and $S_0^{-1}R \simeq \prod_{i=1}^s R_i$ is the direct product of local commutative artinian rings (R_i, \mathfrak{m}_i) where \mathfrak{m}_i is the maximal ideal of the ring $R_i = R_{\mathfrak{p}_i}$, the localization of the ring R at \mathfrak{p}_i . For each non-empty subset I of the set $\{1, \dots, s\}$, let the ring homomorphism $\sigma_I : R \rightarrow R_I := \prod_{i \in I} R_i$ be the composition of the ring monomorphism $\sigma : R \rightarrow S_0^{-1}R$, $r \mapsto r/1$ and the projection $S_0^{-1}R \rightarrow R_I$. The group of units R_I^* of the ring R_I is equal to $\prod_{i \in I} R_i^*$ where $R_i^* = R_i \setminus \mathfrak{m}_i$ is the group of units of the ring R_i . For each number $i = 1, \dots, s$, $\mathfrak{p}'_i := \mathfrak{m}_i \times \prod_{j \neq i} R_j$ is an ideal of the ring $S_0^{-1}R$. Then $\mathfrak{p}''_i := \sigma^{-1}(\mathfrak{p}'_i)$ is an ideal of the ring R .

Proposition 5.10 *Let R be a Noetherian commutative ring, $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the set of its minimal prime ideals, $s \geq 2$ and $\{\mathfrak{p}''_1, \dots, \mathfrak{p}''_s\}$ be as above. Then*

1. $\text{Ass}(R) := \{0\} \cup \{\mathfrak{a}(I) := \bigcap_{i \in I} \mathfrak{p}''_i \mid \emptyset \subsetneq I \subsetneq \{1, \dots, s\}\}$.
2. For each proper subset I of $\{1, \dots, s\}$, $S_{\mathfrak{a}(I)}(R) = \sigma_I^{-1}(R_I^* \times_{R_{CI}}) = R \setminus \bigcup_{i \in I} \mathfrak{p}_i$ and $Q_{\mathfrak{a}(I)}(R) \simeq R_I$ where $CI := \{1, \dots, s\} \setminus I$.
3. $S_0(R) = R \setminus \bigcup_{i=1}^s \mathfrak{p}_i \subseteq S_{\mathfrak{a}}(R)$ for all $\mathfrak{a} \in \text{Ass}(R)$.
4. $S_{\mathfrak{a}(I)}(R) \subseteq S_{\mathfrak{a}(J)}(R)$ iff $I \supseteq J$ where I and J are proper subsets of $\{1, \dots, s\}$.
5. If I and J are proper subsets of $\{1, \dots, s\}$ such that $I \supseteq J$ then, by statement 4 and the universal property of localization, there is the unique ring homomorphism $Q_{\mathfrak{a}(I)}(R) = \prod_{i \in I} R_i \rightarrow Q_{\mathfrak{a}(J)}(R) = \prod_{j \in J} R_j$ which is necessarily the projection onto $\prod_{j \in J} R_j$ in $\prod_{i \in I} R_i$.
6. $\max.\text{Ass}(R) = \{\mathfrak{p}''_i \mid i = 1, \dots, s\}$, $\{Q_{\mathfrak{p}''_i}(R) = R_{\mathfrak{p}_i} \mid i = 1, \dots, s\}$ is the set of maximal localizations of the ring R .

Proof. 1–3. Clearly, $S_0 := S_0(R) = \mathcal{C}_R$ where $\mathcal{C}_R = R \setminus \bigcup_{i=1}^s \mathfrak{p}_i$ is the set of regular elements of the ring R . By Lemma 5.9.(2), $S_0 \subseteq S_{\mathfrak{a}}(R)$ for all $\mathfrak{a} \in \text{Ass}(R)$ and statement 3 follows. We identify the ring R with its image under the ring monomorphisms $R \rightarrow S_0^{-1}R = \prod_{i=1}^s R_i$, $r \mapsto r/1$. By statement 3 and Proposition 3.8.(2), the map $\text{Ass}(R) \rightarrow \text{Ass}(S_0^{-1}R)$, $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$, is a bijection with the inverse map

$$\mathfrak{b} \mapsto R \cap \mathfrak{b}, \quad (26)$$

and the map

$$\{S_{\mathfrak{a}}(R) \mid \mathfrak{a} \in \text{Ass}(R)\} \rightarrow \{S_{\mathfrak{b}}(S_0^{-1}R) \mid \mathfrak{b} \in \text{Ass}(S_0^{-1}R)\}, \quad S_{R \cap \mathfrak{b}}(R) \mapsto S_{\mathfrak{b}}(S_0^{-1}R),$$

is a bijection and

$$S_{R \cap \mathfrak{b}}(R) = \sigma_{R \cap \mathfrak{b}}^{-1}(S_{\mathfrak{b}}(S_0^{-1}R)) \quad (27)$$

where

$$\sigma_{R \cap \mathfrak{b}} : R \rightarrow Q_{R \cap \mathfrak{b}}(R) = S_{R \cap \mathfrak{b}}(R)^{-1}R \simeq S_{\mathfrak{b}}(S_0^{-1}R)^{-1}S_0^{-1}R. \quad (28)$$

Claim: $\text{Ass}(S_0^{-1}R) = \{0\} \cup \{R_I \mid \emptyset \subsetneq I \subsetneq \{1, \dots, s\}\}$. Let \mathcal{R} be the RHS of the equality. Then $\text{Ass}(S_0^{-1}R) \supseteq \mathcal{R}$ since $R_I = \text{ass}(\sigma_{CI}^{-1}(R_{CI}^*))$ where CI is the complement of the proper set I in $\{1, \dots, s\}$. To prove that the reverse inclusion holds, i.e. $\text{Ass}(S_0^{-1}R) \subseteq \mathcal{R}$, we have to show that, for each multiplicatively closed subset S of $S_0^{-1}R$ that does not entirely consists of non-zero-divisors, $\text{ass}(S) = R_I$ for some proper subset I of $\{1, \dots, s\}$. For each element $r \in R$, the subset

of $\{1, \dots, s\}$, $\text{supp}(r) := \{i \mid \pi_i(r) \in R_i^* \text{ and } i \in \{1, \dots, s\}\}$ is called the *support* of the element r where $\pi_i : R \rightarrow S_0^{-1}R = \prod_{j=1}^s R_j \rightarrow R_i$. For all elements $s, t \in S$, $\text{supp}(st) = \text{supp}(s) \cap \text{supp}(t)$. Let $\text{supp}(S) := \bigcap_{s \in S} \text{supp}(s)$. Clearly, there exists an element, say $s \in S$, such that $\text{supp}(s) = \text{Supp}(S)$ (eg, s is the product of all elements of S with all possible distinct supports). Replacing s with s^n for some natural number we may additionally assume that $\pi_i(s) = 0$ for all elements $j \notin I$ (take n such that $\mathfrak{m}_i^n = 0$ for all $i = 1, \dots, s$). Then clearly $\text{ass}(S) = \text{ass}(\{s^i\}_{i \in \mathbb{N}}) = R_{C_{\text{supp}(S)}}$. This finishes the proof of the Claim.

Statement 1 follows from the Claim and (26). For every proper subset I of $\{1, \dots, s\}$, $R \cap R_I = \mathfrak{a}(CI)$, $\sigma_I = \sigma_{R \cap R_{CI}}$ and $S_{R_I}(S_0^{-1}R) = R_{CI}^* \times R_I$. Then, by (27), $S_{R \cap R_I}(R) = \sigma_{R \cap R_I}^{-1}(S_{R_I}(S_0^{-1}R)) = \sigma_{CI}^{-1}(R_{CI}^* \times R_I)$ and, by (28), $Q_{R \cap R_I}(R) \simeq S_{R_I}(S_0^{-1}R)^{-1}(S_0^{-1}R) \simeq R_{CI}$. To finish the proof of statement 2 it remains to show that $S_{R \cap R_I}(R) = R \setminus \bigcup_{i \in I} \mathfrak{p}_i$ for each proper subset I of the set $\{1, \dots, s\}$. An element $s \in R$ belongs to the set $S_{\mathfrak{a}(CI)} = S_{R \cap R_I}(R) = \sigma_{CI}^{-1}(R_{CI}^* \times R_I)$ iff $\sigma(r) \in R_{CI}^* \times R_I$ iff $s \notin \bigcup_{i \in CI} \mathfrak{p}_i$ iff $s \in R \setminus \bigcup_{i \in CI} \mathfrak{p}_i$.

4. Statement 4 follows from statement 2.
5. Statement 5 follows from statement 2.
6. Statement 6 follows from statements 2, 4 and 5. \square

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